Quasi-Discrete Locally Compact Quantum Groups (*)

A. Van Daele (**)

Abstract

Let A be a C*-algebra. Let $A \otimes A$ be the minimal C*-tensor product of A with itself and let $M(A \otimes A)$ be the multiplier algebra of $A \otimes A$. A comultiplication on A is a non-degenerate *-homomorphism $\Delta : A \to M(A \otimes A)$ satisfying the coassociativity law $(\Delta \otimes \iota)\Delta = (\iota \otimes \Delta)\Delta$ where ι is the identity map and where $\Delta \otimes \iota$ and $\iota \otimes \Delta$ are the unique extensions to $M(A \otimes A)$ of the obvious maps on $A \otimes A$. We think of a pair (A, Δ) as a 'locally compact quantum semi-group'.

When these notes where written, in 1993, it was not at all clear what the extra conditions on Δ should be for (A, Δ) to be a 'locally compact quantum group'. This only became clear in 1999 thanks to the work of Kustermans and Vaes. In the compact case however, that is when A has an identity, rather natural conditions can be formulated and so there was a good notion of a 'compact quantum group' already at the time these notes have been written. These compact quantum groups have been studied by Woronowicz.

In these notes, we consider another class of locally compact quantum groups. We assume the existence of a non-zero element h in A such that $\Delta(a)(1 \otimes h) = a \otimes h$ for all $a \in A$. With some extra, but also natural conditions, the element h is unique. We speak of a *quasi-discrete locally compact quantum group*. We also discuss the discrete case and we show that, in that case, there exists such an element h. So, the quasi-discrete case is, at least in principle, more general than the discrete case. Later however, it has been shown by Kustermans that a quasi-discrete locally compact quantum group has to be a discrete quantum group.

We prove the existence of the Haar measure, the regular representation, the fundamental unitary that satifies the Pentagon equation and we obtain the reduced dual.

These notes have not been published. Nevertheless, some of the results and techniques seem to be useful and in recent work, we came across similar settings. Therefore, we have decided to publish these notes in the archive. We have added some comments at the end of the introduction, and also updated the reference list. But apart from these minor changes, the notes are still as they were written in 1993.

August 2004

(*) Lecture Notes K.U.Leuven (June 1993)

(**) Address: Department of Mathematics, K.U.Leuven, Celestijnenlaan 200B, B-3001 Heverlee (Belgium).

0. Introduction

Let us first recall the main ideas behind the C*-approach to quantum groups.

Let G be a locally compact group and let A be the C*-algebra $C_0(G)$ of continuous complex functions on G tending to 0 at infinity. We identify the C*-tensor product $A \overline{\otimes} A$ with $C_0(G \times G)$ and the multiplier algebra $M(A \overline{\otimes} A)$ with the C*-algebra $C_b(G \times G)$ of bounded continuous complex functions on $G \times G$. Then, the product in G gives rise to a *-homomorphism $\Delta : A \to M(A \overline{\otimes} A)$, called the comultiplication. It is defined by $(\Delta(f))(s,t) = f(st)$ when $f \in C_0(G)$ and $s,t \in G$. It is well known that the topological structure of G is completely determined by the C*-algebra structure of A. Moreover, it is easy to see that the map Δ completely determines the group structure.

The idea of the C*-algebra approach to quantum groups is now very natural. The abelian C*-algebra A is replaced by any C*-algebra. So a 'locally compact quantum group' is a pair (A, Δ) where now A is any C*-algebra and Δ a *-homomorphism of A into $M(A \overline{\otimes} A)$ satisfying certain properties.

It is clear that we need extra conditions on Δ. Any abelian C*-algebra A has the form $C_0(G)$ for some locally compact space G but not every *-homomorphism $\Delta : A \to M(A \overline{\otimes} A)$ will come from a group structure on G as above.

Some conditions on Δ are quite natural. First, Δ has to be a non-degenerate *-homomorphism in the sense of [29]. This means that $\Delta(A)(A \otimes A)$ is dense in $A \overline{\otimes} A$. In the abelian case, i.e. when $A \cong C_0(G)$, with G a locally compact space, this will guarantee that Δ comes from a product on G. The coassociativity law $(\Delta \otimes \iota)\Delta = (\iota \otimes \Delta)\Delta$ will give that this procuct is associative. Therefore, a C*-algebra A with a non-degenerate *-homomorphism $\Delta : A \to M(A \overline{\otimes} A)$ satsifying coassociativity can be thought of as a locally compact quantum semi-group.

What extra conditions are needed in order to have a locally compact quantum group? The answer to this question is not yet clear. We want the conditions to be such that most of the theory of locally compact groups can be generalized to the quantum groups. We like to prove the existence of a unique Haar measure, to have a nice representation theory, ... We like to define a dual pair $(\hat{A}, \hat{\Delta})$ that is again a locally compact quantum group (corresponding to the dual group $\hat{G}$ of G if $A = C_0(G)$). We would like to have that (A, Δ) is again the dual pair associated to $(\hat{A}, \hat{\Delta})$, ...

In the compact case, that is when A has an identity, most of this program has been carried out by Woronowicz. He imposed rather natural conditions on Δ to have a 'compact quantum group'. He proved the existence of a Haar measure in this case. He also developed the representation theory, he obtained the dual, ... [30, 31].

On the other hand, the discrete case has been studied by Podleś and Woronowicz in [17]. In their work, the discrete quantum groups are the duals of the compact ones. The discrete case is also studied by Effros and Ruan [6]. But there, the approach is more algebraic. It does not really fit into the C*-algebra framework.

In this paper, we study the discrete case independently. In the philosophy of the C*-algebra approach to quantum groups, discreteness is a property of the topology of the group and therefore, in the quantum group case, we must translate it into a property of the underlying C*-algebra. This is not so difficult, we discuss this in section 2 of this paper.

However, there is also the following simple observation. If G is a discrete group and h is the function δ_e defined as 1 on the identity e and 0 elsewhere, then h is an element in $C_0(G)$ such that $(\Delta(f))(1 \otimes h) = f \otimes h$ for all $f \in C_0(G)$. Indeed, when $s, t \in G$ we have

$$(\Delta(f))(1 \otimes h)(s,t) = f(st)\delta_e(t) = f(s)\delta_e(t) = (f \otimes h)(s,t).$$

We can also take the existence of such an element as an axiom. So we introduce the notion of a quasi-discrete quantum group as a pair (A, Δ) of a C*-algebra A and a comultiplication Δ such that there is a non-zero element $h \in A$ with the property that $\Delta(a)(1 \otimes h) = a \otimes h$ for all $a \in A$. Of course also here we need some extra conditions to distinguish from the semi-group case.

It turns out that in the discrete case, such an element h automatically exists. So, a discrete quantum group is also quasi-discrete. It is not clear however if the second class is really bigger that the first one. We have no examples to show this. Certainly, the techniques that we use are different from the ones that are normally used in the discrete case (where A is a direct sum of full matrix algebras). So, even though the two classes might turn out to be the same, still we have achieved two goals. On the one hand, we have a self-contained treatment of the discrete quantum groups. On the other hand, we have used techniques that will probably be useful to develop the general locally compact quantum groups.

The paper is organized as follows. In *Section* 1, we try to find the natural extra conditions on Δ. We focus on the abelian case here. In *Section* 2 we look at the (proper) discrete case and we compare this with the work of Effros and Ruan. In *Section* 3 we give the precise definition of the quasi-discrete locally compact quantum groups.

In the following sections we develop the theory for our quasi-disrete locally compact quantum groups. The main point is the construction of the antipode S as a linear operator on A. This is done in *Section* 4. We must mention here that this is another main difference with the approach of Effros and Ruan. The existence of an antipode is one of their axioms. In the C*-algebra approach to quantum groups however, it is not so natural to assume the existence of the antipode. It is often a basic difficulty that the antipode is an unbounded, anti-homomorphism. This is also one of the main differences with the earlier von Neumann algebra approach (the Kac algebras) where the antipode was assumed to be bounded, in fact with square one. This older theory was no longer satisfactory since the more recent interesting examples were discovered.

We obtain some nice properties of the antipode that turn out to be very useful for the next sections. In *Section* 5 we use it to construct the Haar measure and in *Section* 6 to obtain the regular representation. In *Section* 7, we construct the fundamental unitary that satisfies the Pentagon equation. There, we also obtain the reduced dual.

In the discrete case, the Haar weight is semi-finite, but in the more general quasi-discrete case, it is not. Perhaps this is not so surprising. After all, in the compact case, the Haar measure need not be faithful. And in a way, these properties are dual to each other. The existence of compact quantum groups with a non-faithful Haar measure therefore may indicate that there are quasi-discrete quantum groups that are not discrete.

In these notes, we will mostly work with separable C*-algebras for technical convenience. We believe though, that this is not essential and that all the arguments can be formulated also in the non-separable case.

June 1993

Note

These lecture notes have been written in 1993. At that time, there was a clear notion of a 'compact quantum group' (see [30, 31]) and about simultaneously with the appearance of these notes, a notion of 'discrete quantum groups' was developed by Effros and Ruan (see [6]). The notion of a locally compact quantum group was fully established in 1999 by Kustermans and Vaes (see [11], [12] and [13]).

In these notes, a certain class of 'locally compact quantum groups' is developed, the so-called quasi-discrete locally compact quantum groups. Shortly after this work was done, it was shown by Kustermans [10], that a quasi-discrete locally compact quantum group was actually a discrete quantum group (as introduced by Effros and Ruan in [6] and later by myself in [26]). For this reason, we decided not to publish these notes. Also the work by Kustermans has not been published.

Still, we believe that this work is of some interest. Recently, we have been involved in some research where a similar setting arose (although purely algebraic), see [14] and [28]. This is the reason why we have decided to publish this paper on the net. We have not made changes to the original version of 1993. Except for the abstract, this note and the references (which have been updated), the paper is the same as in 1993. This has to be taken into account by the reader.

August 2004

Acknowledgements

Most of this work was done while we were visiting the University of Oslo in the summer of 1992. We were greatly inspired by many discussions we had there with E. Alfsen on

the Haar measure (for locally compact groups) and we are grateful to him, both for these discussions and for his hospitality while we were staying in Oslo.

1. Locally compact quantum groups

Consider an abelian C*-algebra A and a *-homomorphism $\Delta : A \to M(A \overline{\otimes} A)$. We know that A has the form $C_0(X)$ where X is a locally compact space. We will look for conditions on Δ to have that it is of the form $(\Delta f)(s,t) = f(st)$ for a multiplication on X that makes X into a locally compact group.

We will not prove very deep results in this section. It will mainly serve as a motivation and we will use the abelian case to illustrate some aspects in the general case. We are interested in finding a set of axioms that will also work in the non-abelian case. So, when dealing here with the abelian case, we must try to avoid the abelianness as much as possible and certainly in the formulation of the conditions.

So, assume that $A = C_0(X)$ where X is a locally compact space. Then, the *-homomorphism $\Delta : A \to M(A \overline{\otimes} A)$ gives a mapping from $C_0(X)$ into $C_b(X \times X)$. The following regularity condition is sufficient to have that Δ is given by a product law on X.

1.1 Proposition. Assume that $\Delta(A)(A \otimes A)$ is dense in $A \overline{\otimes} A$. Then there is a continuous map $(s,t) \in X \times X \to st \in X$ such that $(\Delta f)(s,t) = f(st)$ when $f \in A$ and $s,t \in X$.

Proof : Choose two elements $s,t \in X$ and consider the map $f \to (\Delta f)(s,t)$. It is clear that this is a *-homomorphism from A to $\mathbb{C}$. If $(\Delta f)(s,t) = 0$ for all f, then $((\Delta f)(g \otimes h))(s,t) = 0$ for all $f,g,h \in A$. By the density condition, $f(s)g(t) = (f \otimes g)(s,t) = 0$ for all $f,g \in A$. This is impossible. Hence the above *-homomorphism is non-zero. Therefore, there is an element in X, denoted by st, such that $(\Delta f)(s,t) = f(st)$ for all $f \in A$.

This proves the existence of the product. The continuity follows from the fact that the map $(s,t) \to f(st)$ is continuous for all $f \in A$.

A *-homomorphism $\Delta : A \to M(A \overline{\otimes} A)$ satisfying the condition that $\Delta(A)(A \otimes A)$ is dense in $A \overline{\otimes} A$ is called non-degenerate. It is a morphism from A to $A \overline{\otimes} A$ in the sense of [29].

We now want the product in X to be associative. For this we need the coassociativity of Δ, as formulated in the following proposition.

1.2 Proposition Assume that Δ is non-degenerate. Consider the *-homomorphisms $\Delta \otimes \iota$ and $\iota \otimes \Delta$ on $A \otimes A$ where ι is the identity map. They have unique extensions to *-homomorphisms on $M(A \overline{\otimes} A)$. We denote these extensions still by $\Delta \otimes \iota$ and $\iota \otimes \Delta$. Assume that $(\Delta \otimes \iota)\Delta = (\iota \otimes \Delta)\Delta$. Then, the product on X, defined by Δ as in proposition 1.1, is associative.

Proof : It is not so hard to show that the *-homomorphisms $\Delta \otimes \iota$ and $\iota \otimes \Delta$ have unique extensions. This can be done, also when A is non-abelian. In the abelian case, it is in fact obvious. It is clear that the extensions are given by $((\Delta\otimes\iota)f)(r,s,t) = f(rs,t)$ and $((\iota \otimes \Delta)f)(r,s,t) = f(r,st)$ whenever $f \in C_b(X \times X)$ and $r,s,t \in X$. Then it is also clear that the condition $(\Delta \otimes \iota)\Delta = (\iota \otimes \Delta)\Delta$ gives the associativity of the product. Indeed

$$((\Delta \otimes \iota)\Delta f)(r,s,t) = (\Delta f)(rs,t) = f((rs)t)$$
$$((\iota \otimes \Delta)\Delta f)(r,s,t) = (\Delta f)(r,st) = f(r(st)).$$

These two conditions bring us to the following definitions. Here, A is again any C*-algebra.

1.3 Definition Let A be a C*-algebra and $\Delta : A \to M(A\overline{\otimes}A)$ a *-homomorphism, where $A\overline{\otimes}A$ is some C*-tensor product of A with itself. Assume that Δ is non-degenerate, i.e. that $\Delta(A)(A\otimes A)$ is dense in $A\overline{\otimes}A$. Also assume that Δ is coassociative, i.e. that $(\Delta \otimes \iota)\Delta = (\iota \otimes \Delta)\Delta$. Then Δ is called a comultiplication on A. We will also call the pair (A, Δ) a locally compact quantum semi-group.

We now proceed by looking for conditions that are sufficient, in the abelian case, to have a group. So, in what follows, A is again an abelian C*-algebra $C_0(X)$ and Δ is a comultiplication on A.

The following conditions are quite natural.

1.4 Proposition Assume that $\Delta(A)(1\otimes A)$ and $\Delta(A)(A\otimes 1)$ are dense subspaces of $A\overline{\otimes}A$. Then the product in X, associated with Δ, has the cancellation law, i.e. if $st = rt$ then $s = r$ and if $ts = tr$ then $s = r$.

Proof : Suppose that $r,s,t \in X$ and that $st = rt$. Then, for all $f,g \in A$ we have

$$(\Delta(f)(1 \otimes g))(s,t) = f(st)g(t)$$
$$(\Delta(f)(1 \otimes g))(r,t) = f(rt)g(t)$$

so that

$$(\Delta(f)(1 \otimes g))(s,t) = (\Delta(f)(1 \otimes g))(r,t).$$

By the density of $\Delta(A)(1 \otimes A)$ in $A\overline{\otimes}A$, we have $f(s)g(t) = f(r)g(t)$ for all $f, g \in A$. Hence $r = s$.

Similarly, if $ts = tr$, the density of $\Delta(A)(A \otimes 1)$ will give $s = r$.

We want to make a remark here about the relation of these conditions and the previous ones. First it is clear that the density of $\Delta(A)(1 \otimes A)$ in $A\overline{\otimes}A$ automatically gives the density of $\Delta(A)(A \otimes A)$ in $A\overline{\otimes}A$, and similarly for $\Delta(A)(A \otimes 1)$. A second remark is that one can now formulate the coassociativity rule, as

$$(a \otimes 1)(\Delta \otimes \iota)(\Delta(b)(1 \otimes c)) = (\iota \otimes \Delta)((a \otimes 1)\Delta(b))(1 \otimes c),$$

that is, without having the need to consider the extensions of $\Delta \otimes \iota$ and $\iota \otimes \Delta$ to the multiplier algebra (see e.g. [24]).

Now, it is known that a compact semi-group with cancellation is actually a group, see [8]. We give here, for completeness, a simple proof of this result.

1.5 Propostion Let G be a compact semi-group with cancellation, then G is a group.

Proof : Take any element $s \in G$. Consider the closed semi-group H generated by s. Consider the family of all closed ideals in H. The intersection of two ideals is non-empty and again an ideal. Because H is compact, the intersection of all ideals is an ideal I. It is the minimal ideal in H. Because H is abelian, tI is again an ideal for all t in H. It is contained in I and by minimality, it is equal to I. So, $tI = I$ for all $t \in H$.

Consider any element $t \in I$ and choose $e \in I$ scuh that $te = t$. Multiply with any element $r \in G$ to the right and cancel t to get $er = r$. Then multiply with any element $t \in G$ to the left and cancel r to get $te = t$. So, e is the identity in G.

Now, from $se = s$ and the fact that $e \in I$ and that I is an ideal, we must have $s \in I$. Because $sI = I$ and $e \in I$, s has an inverse. So G is a group.

So, if A is any abelian C*-algebra with an identity and if Δ is a comultiplication on A such that $\Delta(A)(1 \otimes A)$ and $\Delta(A)(A \otimes 1)$ are dense subspaces of $A \overline{\otimes} A$, then $A \cong C_0(G)$ where G is a compact group and Δ is given by the multiplication on G as before.

So we are lead naturally to the following definition, due to Woronowicz [31] :

1.6 Definition If A is any C*-algebra with identity and Δ a comultiplication on A, such that the sets $\Delta(A)(1 \otimes A)$ and $\Delta(A)(A \otimes 1)$ are dense in $A \overline{\otimes} A$, then (A, Δ) is called a compact quantum group.

Woronowicz proved the existence of the Haar measure for such compact quantum groups when A admits a faithful state, see [31]. We adapted his proof to obtain the Haar measure without this restriction on A, see [25]. For compact quantum groups, the Haar measure is unique but, in general, it need not be faithful. Nevertheless, much of the theory of compact groups extends to the compact quantum groups as defined above, see [30, 31].

In the non-compact case, the above conditions are not sufficient to have a group. We illustrate this with some simple examples.

1.7 Examples i) Consider the (discrete) semi-group $G = \mathbb{N} \cup \{\infty\}$ (with $n + \infty = \infty + n = \infty$ for all $n \in \mathbb{N}$). Let δ_p denote the function that is 1 on p and 0 elsewhere. We have that $(\Delta\delta_\infty)(n, m) = \delta_\infty(n + m)$. Now $n + m = \infty$ if and only if $n = \infty$ or $m = \infty$. So we get

$$\Delta(\delta_\infty) = \delta_\infty \otimes 1 + 1 \otimes \delta_\infty - \delta_\infty \otimes \delta_\infty.$$

In particular $\Delta(\delta_\infty)(\delta_\infty \otimes 1) = \delta_\infty \otimes 1$ and we see that in this example $\Delta(A)(A \otimes 1)$ is not a subspace of $A \overline{\otimes} A$. Of course, also here, we do not have cancellation.

ii) Let $G = \mathbb{N}$. For any pair (n, m) in $\mathbb{N}$ we have

$$\Delta(\delta_n)(1 \otimes \delta_m) = \begin{cases} \delta_{n-m} \otimes \delta_m & \text{if } n \geq m \\ 0 & \text{if } n < m. \end{cases}$$

In particular, $\Delta(A)(1 \otimes A)$ is a dense subspace of $A \overline{\otimes} A$ here. Similarly of course $\Delta(A)(A \otimes 1)$ is a dense subspace of $A \overline{\otimes} A$. Indeed, $\mathbb{N}$ has the cancellation property. Still, $\mathbb{N}$ is not a group.

In the last example, we see that there are elements $a, b \in A$ such that $\Delta(a)(1 \otimes b) = 0$ with $a \neq 0$ and $b \neq 0$. This cannot happen in the group case : if $f(st)g(t) = 0$ for all $s, t \in G$, then $f(r)g(t) = 0$ for all $r, t \in G$ and hence $f = 0$ or $g = 0$.

It turns out that this is precisely the extra condition that we need in general.

1.8 Theorem Let A be an abelian C*-algebra with a comultiplication Δ such that $\Delta(A)(1 \otimes A)$ and $\Delta(A)(A \otimes 1)$ are dense subspaces of $A \overline{\otimes} A$. Also assume that $\Delta(a)(1 \otimes b) = 0$ with $a, b \in A$ implies $a = 0$ or $b = 0$. Then $A \cong C_0(G)$ where G is a locally compact group and Δ is given by the multiplication on G as before.

Proof : We know already that $A \cong C_0(G)$ and that G is a semi-group with the cancellation property. Let us now see what the last injectivity condition means in terms of G.

Choose any non-empty open subset V of G. We claim that the set $\{rs \mid r \in G, s \in V\}$ is dense in G. Indeed, suppose that it is not dense. Then there is an element $f \in C_0(G)$ such that $f \neq 0$ but $f(rs) = 0$ for all $r \in G$ and all $s \in V$. Choose an element $g \in C_0(G)$ such that $g \neq 0$ and such that g has support in V. Then $f(rs)g(s) = 0$ for all $r, s \in G$. Hence $(\Delta f)(1 \otimes g) = 0$. This would imply $f = 0$ or $g = 0$ by assumption. So we get a contradiction.

Now, let $s, t \in G$. For any pair V, W of open neighbourhoods of s and t respectively, we have, by the above property, a point r in G such that $rs = t$. Consider the pairs (V, W) as an index set I, ordered by the opposite inclusion. Then we find nets $(r_\alpha), (s_\alpha), (t_\alpha)$ such that $r_\alpha s_\alpha = t_\alpha$ for all $\alpha \in I$ and such that $s_\alpha \to s$ and $t_\alpha \to t$.

For every $f, g \in A$ we have

$$((\Delta f)(1 \otimes g))(r_\alpha, s_\alpha) = f(r_\alpha s_\alpha)g(s_\alpha) = f(t_\alpha)g(s_\alpha)$$

and this converges to $f(t)g(s)$. Choose f and g such that $f(t)g(s) \neq 0$. Let K_1, K_2 be compact sets in G such that $(\Delta f)(1 \otimes g)$ is strictly smaller than $\frac{1}{2}|f(t)g(s)|$ outside $K_1 \times K_2$. Then, for α large enough, we will have that $|((\Delta f)(1 \otimes g))(r_\alpha, s_\alpha)| \geq$

$\frac{1}{2}|f(t)g(s)|$ and so $r_\alpha \in K_1$ and $s_\alpha \in K_2$. Then, we can find a subnet of (r_α) that converges to a point r. So in the limit, we will get $rs = t$.

So for every two elements $s, t \in G$ we find $r \in G$ such that $rs = t$. Given $s \in G$ we get $e \in G$ such that $es = s$. As in the proof of 1.5, we see, using cancellation that e is an identity. Again, given s we find r such that $rs = e$. Multiplying with s on the left and cancelling s on the right, we get $sr = e$. So every element has an inverse.

We also must show that the maps $s \to s^{-1}$ is continuous. For all $f, g \in A$ we have that $((\Delta f)(1 \otimes g))(s^{-1}, s) = f(s^{-1}s)g(s) = f(e)g(s)$ so that $s \to ((\Delta f)(1 \otimes g))(s^{-1}, s)$ is continuous. By density of $\Delta(A)(1 \otimes A)$ in $A \overline{\otimes} A$, still $s \to f(s^{-1})g(s)$ will be continuous for all $f, g \in A$. Hence, $s \to s^{-1}$ is continuous.

It is clear that the condition $\Delta(a)(1 \otimes b) = 0$ implies $a = 0$ or $b = 0$ can be replaced by the similar condition $\Delta(a)(b \otimes 1) = 0$ implies $a = 0$ or $b = 0$. But there is more. If we examine the proof of the previous theorem, we see that we also have the following result.

1.9 Theorem Let A be an abelian C*-algebra with a comultiplication Δ such that $\Delta(A)(1 \otimes A)$ and $\Delta(A)(A \otimes 1)$ are subspaces of $A \overline{\otimes} A$. Assume further that

$$\Delta(a)(1 \otimes b) = 0 \Rightarrow a \otimes b = 0$$
$$\Delta(a)(b \otimes 1) = 0 \Rightarrow a \otimes b = 0.$$

Then $A \cong C_0(G)$ where G is a locally compact group and Δ is given by the multiplication on G.

Proof : The injectivity rules give that, for any pair $s, t \in G$, there exists elements u and v such that $su = t$ and $vs = t$. This was shown in the previous theorem. Remark that we need that $\Delta(A)(1 \otimes A)$ and $\Delta(A)(A \otimes 1)$ are subsets of $A \overline{\otimes} A$. We do not need the density of these spaces. So $tG = Gt = G$ for all $t \in G$. Then G is a group.

The continuity of $s \to s^{-1}$ is proven as in the previous case.

We finish this section with some important remarks.

1.10 Remarks i) Consider the linear map $T : A \otimes A \to A \overline{\otimes} A$ defined by $T(a \otimes b) = \Delta(a)(1 \otimes b)$. In the group case $A = C_0(G)$, this map extends to an isomorphism on $A \overline{\otimes} A$ given by $(Tf)(s, t) = f(st, t)$. The inverse is of course given by $(T^{-1}f)(s, t) = f(st^{-1}, t)$.

ii) We see in the above theorems that the injectivity rule $T(a \otimes b) = 0 \Rightarrow a \otimes b = 0$ implies that also the extension of T remains injective. A similar result is also true for the compact quantum groups (see [31]). And we see further in this paper (see proposition 3.9 below), that it is also true for the quasi-discrete quantum groups.

iii) In our paper on multiplier Hopf algebras, we needed the fact that T was a bijection between the algebraic tensor product $A \otimes A$ and itself. This, together with

the property $\Delta(A)(A\otimes 1) = A\otimes A$ was enough, in that algebraic context, to develop the theory ([24]).

In view of all these remarks, for a pair (A, Δ) with any C*-algebra A and a comultiplication Δ on A to be a locally compact quantum group, we will need conditions like the density of $\Delta(A)(A\otimes 1)$ and $\Delta(A)(1\otimes A)$ in $A\overline{\otimes}A$ and the injectivity of the maps $a\otimes b \to \Delta(a)(1\otimes b)$ and $a\otimes b \to \Delta(a)(b\otimes 1)$.

2. Discrete locally compact quantum groups

A discrete locally compact quantum group is, in the above philosophy, a locally compact quantum group where the underlying space is discrete. Therefore, we must look at properties of $C_0(X)$ when X is discrete, and see how they can be generalized to a not necessarily abelian C*-algebra.

If $A = C_0(X)$, then the multiplier algebra $M(A)$ of A can be identified with the C*-algebra $C_b(X)$ of all bounded continuous complex functions on X. If X is discrete, then $C_b(X)$ consists of all bounded complex functions on X and therefore coincides with the bidual of $C_0(X)$. We can take this property as the basis for the following definition.

2.1 Definition A C*-algebra A such that the multiplier algebra $M(A)$ coincides with the full bidual A^{**} is called a discrete quantum space.

Recall that A^{**} coincides with the von Neumann algebra A'' when A is considered in its universal representation [16] and that then, $M(A)$ is the subalgebra of elements x in A'' such that xa and ax are elements in A for all $a \in A$. So A satisfies the above property if $xa \in A$ and $ax \in A$ whenever $a \in A$ and $x \in A''$.

It is not hard to see that such a C*-algebra must be a direct sum of components that are either full matrix algebras or C*-algebras of compact operators on infinite-dimensional Hilbert spaces (see e.g. [15]).

So let $A = \sum_\alpha \oplus A_\alpha$ where each A_α is of the form $\mathcal{K}(\mathcal{H}_\alpha)$, the C*-algebra of compact operators on some Hilbert space $\mathcal{H}_\alpha$. Here $\mathcal{H}_\alpha$ can be finite-dimensional so that $\mathcal{K}(\mathcal{H}_\alpha) = M_n(\mathbb{C})$ where n is the dimension of $\mathcal{H}_\alpha$. Recall that the direct sum $\sum_\alpha \oplus A_\alpha$ consists of elements $(x_\alpha)_\alpha$ where $x_\alpha \in A_\alpha$ for each α and $\|x_\alpha\| \to 0$ as $\alpha \to \infty$ (the index set being considered as a discrete space here).

We have $A\overline{\otimes}A = \sum_{\alpha,\beta} \oplus(A_\alpha \otimes A_\beta)$ and $M(A\overline{\otimes}A) = \prod_{\alpha,\beta} B(\mathcal{H}_\alpha \otimes \mathcal{H}_\beta)$, where $B(\mathcal{H}_\alpha \otimes \mathcal{H}_\beta)$ denotes the C*-algebra of all bounded linear operators on $\mathcal{H}_\alpha \otimes \mathcal{H}_\beta$. Recall also that the product $\prod_\alpha A_\alpha$ consists of elements $(x_\alpha)_\alpha$ such that $x_\alpha \in A_\alpha$ for all α and such that $\|x_\alpha\|$ is bounded in α.

Now, let $\Delta : A \to M(A \overline{\otimes} A)$ be a comultiplication on A. In this case, it gives, in a natural way, a family of *-homomorphisms $\Delta_\alpha^{\beta\gamma} : A_\alpha \to B(\mathcal{H}_\beta \otimes \mathcal{H}_\gamma)$. Then, we have the following result (see also [6] and [17]).

2.2 Proposition Assume $\Delta(A)(A \otimes 1) \subseteq A \overline{\otimes} A$ and $\Delta(A)(1 \otimes A) \subseteq A \overline{\otimes} A$. Then given α, β, there exists only finitely many γ such that $\Delta_\alpha^{\beta\gamma} \neq 0$. Similarly, given α, γ, there exists only finitely many β such that $\Delta_\alpha^{\beta\gamma} \neq 0$.

Proof : Denote by e_α the identity in A_α for any index α. Fix α and β. Then $\Delta(e_\alpha)(e_\beta \otimes 1)$ is a projection in $A \overline{\otimes} A$ by assumption. It must lie in $\sum_\gamma \oplus (A_\beta \otimes A_\gamma)$. Such a projection must be of the form $(x_\gamma)_\gamma$ with $x_\gamma \in A_\beta \otimes A_\gamma$ and x_γ a projection for all γ. Because $\|x_\gamma\| \to 0$ as $\gamma \to \infty$, we must have $x_\gamma = 0$ except for a finite number of indices γ. This proves the first statement. Similarly for the other one.

Now assume that all A_α are finite-dimensional. Consider the ideal $\mathcal{A}$ of A consisting of all elements (x_α) with only finitely many x_α non-zero. It is clear from the above result that $\Delta(\mathcal{A})(\mathcal{A} \otimes 1)$ and $\Delta(\mathcal{A})(1 \otimes \mathcal{A})$ now are subspaces of the algebraic tensor product $\mathcal{A} \otimes \mathcal{A}$ of $\mathcal{A}$ with itself. If the linear maps T_1 and T_2, defined on $\mathcal{A} \otimes \mathcal{A}$ by $T_1(a \otimes b) = \Delta(a)(1 \otimes b)$ and $T_2(a \otimes b) = \Delta(a)(b \otimes 1)$ are bijective, then the pair $(\mathcal{A}, \Delta)$ is a multiplier Hopf *-algebra (in the sense of [24]).

We propose the following definition for a discrete quantum group.

2.3 Definition A discrete quantum group is a pair (A, Δ) of a C*-algebra A and a comultiplication Δ on A where A has the form $\sum_\alpha \oplus A_\alpha$ with each A_α a full matrix algebra and where the linear maps T_1 and T_2, defined by $T_1(a \otimes b) = \Delta(a)(1 \otimes b)$ and $T_2(a \otimes b) = \Delta(a)(b \otimes 1)$, give bijections of the algebraic tensor product $\mathcal{A} \otimes \mathcal{A}$ of the *-algebra $\mathcal{A}$ of elements (x_α) in A with only finitely many x_α non-zero.

Essentially, we get a discrete quantum group when we have a multiplier Hopf *-algebra $(\mathcal{A}, \Delta)$ with a *-algebra $\mathcal{A}$ as above. We also know from the theory of multiplier Hopf algebras that there exists a unique counit ϵ and an antipode S which is invertible [24]. So we get the same objects as Effros and Ruan in [6]. The fact of working with C*-algebras is no longer essential.

Given the counit ϵ, we have the following result.

2.4 Proposition Let (A, Δ) be a discrete quantum group. Let ϵ be the counit. Then there is a self-adjoint projection h in A such that $ha = ah = \epsilon(a)h$ for all $a \in A$. We also have $\Delta(a)(1 \otimes h) = a \otimes h$ and $\Delta(a)(h \otimes 1) = h \otimes a$ for all $a \in A$.

Proof : Since ϵ is a *-homomorphism from $\mathcal{A}$ to $\mathbb{C}$, the kernel of ϵ is a two sided ideal in $\mathcal{A}$ with codimension 1. Hence one of the direct summands of A must be one-dimensional. Let h be the identity of this component. Then h is a self-adjoint projection in A such that $ha = ah = \epsilon(a)h$.

Clearly we have also

$$\begin{aligned}\Delta(a)(1 \otimes h) &= (\iota \otimes \epsilon)\Delta(a) \otimes h = a \otimes h \\ \Delta(a)(h \otimes 1) &= h \otimes (\epsilon \otimes \iota)\Delta(a) = h \otimes a.\end{aligned}$$

In the case of a discrete group, the element h is of course the function δ_e where e is the identity of the group.

As we explained already in the introduction, we will take the existence of such an element h as an axiom. This will give us the notion of a quasi-discrete quantum group.

By proposition 2.4, we will have that a discrete quantum group is automatically quasi-discrete. So, all the results that we obtain in this paper for quasi-discrete quantum groups are also valid for the discrete ones. In the next sections, where appropriate, we will indicate the extra results that can be obtained in the proper discrete case.

3. Quasi-discrete quantum groups

Let us start with a pair (A, Δ) where A is a C*-algebra and Δ a comultiplication on A. We make the following assumption :

3.1 Assumption Assume that there is a non-zero element h in A such that $\Delta(a)(1 \otimes h) = a \otimes h$ for all $a \in A$.

If we multiply the above equation to the right with h^*, we see immediately that we can assume that $h \geq 0$.

If we also assume that $\Delta(A)(A \otimes 1)$ is a dense subspace of $A \overline{\otimes} A$, we can prove a number of results.

3.2 Lemma There exists a non-zero homomorphism $\epsilon : A \to \mathbb{C}$ such that $ah = \epsilon(a)h$ for all $a \in A$.

Proof : For all $b \in A$ we have

$$(\Delta(a)(b \otimes 1))(1 \otimes h) = ab \otimes h.$$

By the density of $\Delta(A)(A \otimes 1)$ in $A \overline{\otimes} A$, it follows that

$$(A \otimes A)(1 \otimes h) \subseteq A \otimes h$$

and so $Ah \subseteq \mathbb{C}h$. This implies the existence of a linear map $\epsilon : A \to \mathbb{C}$ given by $ah = \epsilon(a)h$. Clearly

$$\epsilon(ab)h = abh = a\epsilon(b)h = \epsilon(b)\epsilon(a)h$$

so that ϵ is a homomorphism. Also $h^*h = \epsilon(h^*)h$ and because $h \neq 0$ we must have $\epsilon(h^*) \neq 0$. So ϵ is non-zero.

3.3 Lemma We can choose h such that $h^2 = h = h^*$.

Proof : We have $h^*h = \epsilon(h^*)h$. For any $\lambda \in \mathbb{C}$ we get $(\lambda h)^*(\lambda h) = \bar{\lambda}\epsilon(h^*)(\lambda h)$. So, if $\bar{\lambda}\epsilon(h^*) = 1$, and if we replace h by λh, we find an element h, still satisfying assumption 3.1 but now also $h^*h = h$. Hence $h = h^*$ and $h^2 = h$.

From now on, we make this choice of h. In particular $\epsilon(h) = 1$.

3.4 Lemma ϵ is a *-homomorphism.

Proof For all $a \in A$ we have

$$\begin{aligned} ha^*h &= h\epsilon(a^*)h = \epsilon(a^*)h \\ ha^*h &= (ah)^*h = \overline{\epsilon(a)}h \end{aligned}$$

so that $\epsilon(a^*) = \overline{\epsilon(a)}$.

3.5 Lemma For all $a \in A$ we have

$$\begin{aligned} (\iota \otimes \epsilon)\Delta(a) &= a \\ (\epsilon \otimes \iota)\Delta(a) &= a. \end{aligned}$$

Proof : We have

$$a \otimes h = \Delta(a)(1 \otimes h) = (\iota \otimes \epsilon)\Delta(a) \otimes h$$

so that $(\iota \otimes \epsilon)\Delta(a) = a$.

To prove the second formula, let $a, b \in A$. Then

$$\begin{aligned} ((\Delta \otimes \iota)(a \otimes b))(1 \otimes h \otimes 1) &= (\Delta(a) \otimes b)(1 \otimes h \otimes 1) \\ &= (\Delta(a)(1 \otimes h)) \otimes b = a \otimes h \otimes b. \end{aligned}$$

This last element is $\sigma_{23}(a \otimes b \otimes h)$ where σ_{23} flips the last two factors.

If we replace $a \otimes b$ by $\Delta(a)$, we get

$$\begin{aligned} \sigma_{23}(\Delta(a) \otimes h) &= ((\Delta \otimes \iota)\Delta(a))(1 \otimes h \otimes 1) \\ &= ((\iota \otimes \Delta)\Delta(a))(1 \otimes h \otimes 1). \end{aligned}$$

Now, we multiply with c in the first factor of the tensor product. Then we obtain

$$\sigma_{23}((\Delta(a)(c \otimes 1)) \otimes h) = (\iota \otimes \Delta)(\Delta(a)(c \otimes 1))(1 \otimes h \otimes 1).$$

By density of $\Delta(A)(A \otimes 1)$ in $A \overline{\otimes} A$, we can replace $\Delta(a)(c \otimes 1)$ by $a \otimes b$ and we obtain

$$\sigma_{23}(a \otimes b \otimes h) = (a \otimes \Delta(b))(1 \otimes h \otimes 1).$$

Hence

$$h \otimes b = \Delta(b)(h \otimes 1) = h \otimes (\epsilon \otimes \iota)\Delta(b)$$

and $b = (\epsilon \otimes \iota)\Delta(b)$.

Remark that we find that also $\Delta(a)(h \otimes 1) = h \otimes a$ and not only $\Delta(a)(1 \otimes h) = a \otimes h$.

From the formulas in 3.5 we also get the uniqueness of h.

3.6 Lemma h is unique.

Proof : Assume that h and h' are non-zero elements satisfying $\Delta(a)(1 \otimes h) = a \otimes h$ and $\Delta(a)(1 \otimes h') = a \otimes h'$. Assume that $h^2 = h = h^*$ and similarly for h'. Let ϵ and ϵ' be the associated *-homomorphisms. For all a we have

$$\epsilon(a) = \epsilon((\iota \otimes \epsilon')\Delta(a)) = \epsilon'((\epsilon \otimes \iota)\Delta(a)) = \epsilon'(a).$$

So $\epsilon = \epsilon'$. Then $h = \epsilon(h')h = h'h = \epsilon(h)h' = h'$.

We summarise in the following proposition.

3.7 Proposition Let A be a C*-algebra and Δ a comultiplication on A. Assume that $\Delta(A)(A \otimes 1)$ is a dense subspace in $A \overline{\otimes} A$. Assume there is a non-zero element h in A such that $\Delta(a)(1 \otimes h) = a \otimes h$ for all a. Then there is a unique non-zero self-adjoint projection h in A such that $\Delta(a)(1 \otimes h) = a \otimes h$. We also have $\Delta(a)(h \otimes 1) = h \otimes a$. There is also a unique *-homomorphism $\epsilon : A \to \mathbb{C}$ such that $(\iota \otimes \epsilon)\Delta(a) = a$ and $(\epsilon \otimes \iota)\Delta(a) = a$ for all a. Furthermore $ah = ha = \epsilon(a)h$ for all $a \in A$.

Remark that the statement about the uniqueness of ϵ was more or less proved in the proof of lemma 3.6. Indeed, if ϵ and ϵ' are linear maps such that $(\iota \otimes \epsilon')\Delta(a) = a$ and $(\epsilon \otimes \iota)\Delta(a) = a$ for all a, then $\epsilon = \epsilon'$.

We also remark that the condition $\Delta(A)(A \otimes 1)$ dense in $A \overline{\otimes} A$ is not really necessary, a weaker condition like

$$\{(\omega \otimes \iota)\Delta(a) \mid \omega \in A^*, a \in A\}$$

is in A and spans a dense subspace of A would be sufficient to carry out the above proofs. Having this condition and the existence of h, we see that we have more, namely

$$\{(\epsilon \otimes \iota)\Delta(a) \mid a \in A\} = A$$
$$\{(\iota \otimes \epsilon)\Delta(a) \mid a \in A\} = A.$$

In fact, the condition

$$\{(\iota \otimes \omega)\Delta(a) \mid \omega \in A^*, a \in A\} = A$$

is already a consequence of the existence of h alone. Indeed, if $a \in A$ and $\varphi \in A^*$ such that $\varphi(h) = 1$, then, with $\omega = \varphi(\cdot h)$ we have

$$(\iota \otimes \omega)\Delta(a) = (\iota \otimes \varphi)(\Delta(a)(1 \otimes h)) = a\varphi(h) = a$$

We can illustrate some of the above results with (easy) examples.

3.8 Examples i) If A is a C*-algebra and $\Delta(a) = a \otimes 1$ for all a. Then Δ is a comultiplication. Any element h satisfies $\Delta(a)(1 \otimes h) = a \otimes h$. For any linear functional ϵ with $\epsilon(1) = 1$ we have $(\iota \otimes \epsilon)\Delta(a) = a$. But here $\Delta(A)(A \otimes 1) = A \otimes 1$ and this is not a dense subspace of $A \overline{\otimes} A$.

ii) If $\Delta(a) = 1 \otimes a$, there is no such element h while $\Delta(A)(A \otimes 1)$ is dense in $A \overline{\otimes} A$.

Now let A be a C*-algebra with a comultiplication Δ and assume that $\Delta(A)(A \otimes 1)$ and $\Delta(A)(1 \otimes A)$ are dense subspaces of $A \overline{\otimes} A$. We have not yet used the second density condition, but we will need it later. We know, from our discussion on the abelian case in section 1 that these two density conditions are quite natural.

Assume further that there exists an element h like before, i.e. a self-adjoint projection h satisfying $\Delta(a)(1 \otimes h) = a \otimes h$ and $\Delta(a)(h \otimes 1) = h \otimes a$ for all a. Then we have the existence of a counit ϵ, i.e. a *-homomorphism $\epsilon : A \to \mathbb{C}$ such that $(\iota \otimes \epsilon)\Delta(a) = (\epsilon \otimes \iota)\Delta(a) = a$ for all a. And we have $ah = ha = \epsilon(a)h$.

In section 1, we also saw that, in the abelian case, we need some extra condition to distinguish the group case from the case of a semi-group with cancellation. The condition that we needed was $\Delta(a)(1 \otimes b) = 0 \Rightarrow a \otimes b = 0$.

Here we have the following interesting property.

3.9 Proposition Consider the linear map $T_1 : A \otimes A \to A \overline{\otimes} A$ defined by $T_1(a \otimes b) = \Delta(a)(1 \otimes b)$. If $\Delta(h)(1 \otimes a) = 0$ implies $a = 0$ then this map is injective.

Proof : We know that $(1 \otimes h)\Delta(a) = a \otimes h$. If we apply $\iota \otimes \Delta$ and multiply with $1 \otimes 1 \otimes b$, we get

$$(1 \otimes \Delta(h))\Delta^{(2)}(a)(1 \otimes 1 \otimes b) = (a \otimes \Delta(h))(1 \otimes 1 \otimes b)$$

where we use the notation $\Delta^{(2)}(a)$ for $(\Delta \otimes \iota)\Delta(a)$ as usual. This can be rewritten as

$$(1 \otimes \Delta(h))(\Delta \otimes \iota)(T_1(a \otimes b)) = (1 \otimes \Delta(h))\sigma_{12}(1 \otimes a \otimes b).$$

By linearity we have for all $x \in A \otimes A$,

$$(1 \otimes \Delta(h))(\Delta \otimes \iota)(T_1(x)) = (1 \otimes \Delta(h))\sigma_{12}(1 \otimes x).$$

If now $T_1(x) = 0$ then $(1 \otimes \Delta(h))\sigma_{12}(1 \otimes x) = 0$. If we apply $\omega \otimes \iota \otimes \iota$ with $\omega \in A^*$, we find $\Delta(h)(1 \otimes y) = 0$ with $y = (\omega \otimes \iota)x$. By assumption $y = 0$. Because this is true for all ω, we get also that $x = 0$. Hence T_1 is injective.

We see from the above proof that, if T_1 can be closed (as a linear map from $A\overline{\otimes}A$ to $A\overline{\otimes}A$), then this closure will still be injective. In any case, we see that, if (x_n) is a sequence in $A \otimes A$ such that $T_1 x_n \to 0$ and such that (x_n) converges, then $x_n \to 0$. So, in fact, the inverse of T_1 is closable.

All this leads us to the main object in this paper.

3.10 Definition Let A be a C*-algebra with a comultiplication Δ. Assume that $\Delta(A)(1\otimes A)$ and $\Delta(A)(A\otimes 1)$ are dense subspaces of $A\overline{\otimes}A$. Assume that A has a projection h as before and that $\Delta(h)(1\otimes a) = 0$ implies $a = 0$. Then we call (A, Δ) a quasi-discrete quantum group.

We will see later (in proposition 3.17 below), that it will follow automatically that also $\Delta(h)(a \otimes 1) = 0$ implies $a = 0$. This in turn, will give that the map $T_2 : A \otimes A \to A\overline{\otimes}A$, defined as before by $T_2(a \otimes b) = \Delta(a)(b \otimes 1)$ is injective. The proof is similar as the one for T_1. But it also follows by symmetry. So the two maps T_1 and T_2 are injective (compare with [24]). Therefore, we also get that a quasi-discrete quantum group, where the underlying C*-algebra is a direct sum of full matrix algebras, is a discrete quantum group.

In the remaining of this section, we introduce some subsets of A, canonically associated to such a quasi-discrete quantum group. We also prove some more properties of the element h.

3.11 Notation Let

$$I_0 = \{(\omega \otimes \iota)\Delta(h) \mid \omega \in A^*\}$$
$$J_0 = \{(\iota \otimes \omega)\Delta(h) \mid \omega \in A^*\}.$$

Let also $I = \overline{I_0}$ and $J = \overline{J_0}$.

Remark that any element $\omega \in A^*$ is of the form $\omega_1(\,\cdot\, a)$ (see e.g. [21]). Hence, by our assumptions, I_0 and J_0 are subsets of A.

We will show that $I = J$ and and that J is a closed two-sided ideal. We need a few more properties of h before we can do this. These properties will also be used later.

3.12 Proposition $\Delta(h)(A \otimes 1) \subseteq \Delta(h)(1 \otimes A)^-$

Proof : Let $a \in A$. Because $h^2 = h$ we have,

$$\Delta(h)(a \otimes 1) = \Delta(h)\Delta(h)(a \otimes 1).$$

Since $\Delta(h)(a\otimes 1) \in A\overline{\otimes}A$ and because $\Delta(A)(1\otimes A)$ is dense in $A\overline{\otimes}A$, we can approximate $\Delta(h)(a\otimes 1)$ by finite linear combinations of elements of the form $\Delta(c)(1\otimes d)$. Now

$$\Delta(h)\Delta(c)(1 \otimes d) = \Delta(hc)(1 \otimes d) = \Delta(h)(1 \otimes \epsilon(c)d).$$

So we see that

$$\Delta(h)(a \otimes 1) = \lim_n \Delta(h)(1 \otimes b_n)$$

for some sequence (b_n) in A.

3.13 Proposition I and J are two-sided ideals.

Proof : Given a in A, we have by the previous result that

$$\Delta(h)(a \otimes 1) = \lim_n \Delta(h)(1 \otimes b_n)$$

for some sequence (b_n) in A. If we apply $\iota \otimes \omega$ with $\omega \in A^*$ we find

$$((\iota \otimes \omega)\Delta(h))a = \lim_n (\iota \otimes \omega_n)\Delta(h)$$

where $\omega_n = \omega(\,\cdot\, b_n)$ for all n. This shows that $J_0 A \subseteq J$. Hence $JA \subseteq J$. Furthermore J_0 and J are clearly self-adjoint. A similar argument works for I.

The injectivity condition gives the following property of the ideal I.

3.14 Proposition The ideal I is essential, i.e. if $a \in A$ and $ba = 0$ for all $b \in I$, then $a = 0$.

Proof : If $ba = 0$ for all $b \in I$ then

$$(\omega \otimes \iota)(\Delta(h)(1 \otimes a)) = ((\omega \otimes \iota)\Delta(h)) \cdot a = 0$$

for all $\omega \in A^*$. So $\Delta(h)(1 \otimes a) = 0$ and by assumption $a = 0$.

This guarantees that the ideal I carries enough information. In section 5, where we discuss the Haar measure, we will come back to the fact that possibly I is strictly smaller than A.

In the discrete case however, the fact that I is essential implies that it is all of A. Moreover, the ideal $\mathcal{A}$ that we defined before, consisting of the elements (x_α) with only finitely many x_α non-zero, must be contained in I_0. This can be seen as follows. Take any α and consider the set $(\omega \otimes \iota)\Delta(h)$ with $\omega \in A$ but with support in A_α. As in the proof of proposition 3.13, this is also an ideal. It must be contained in $\mathcal{A}$. Hence, it must be a direct sum of a finite number of summands of A. We must get all components this way because the set I_0 is dense. So we find $\mathcal{A} \subseteq I_0$.

We now turn back to the general quasi-discrete case. We will show that $I = J$. To prove this, we first need some more results on h.

3.15 Proposition We have

$$\Delta_{23}(h)\Delta_{14}(h) = \Delta_{23}(h)\Delta^{(3)}(h),$$

where we use the 'leg numbering' notation (e.g. $\Delta_{23}(h) = 1 \otimes \Delta(h) \otimes 1$). We also use $\Delta^{(3)}(h)$ for $(\Delta \otimes \iota \otimes \iota)(\Delta \otimes \iota)\Delta(h)$ as usual.

Proof :

$$\begin{aligned}\Delta_{23}(h)\Delta^{(3)}(h) &= (1 \otimes \Delta(h) \otimes 1)(\iota \otimes \Delta \otimes \iota)(\Delta^{(2)}(h)) \\ &= (\iota \otimes \Delta \otimes \iota)((1 \otimes h \otimes 1)\Delta^{(2)}(h)).\end{aligned}$$

Now $(1 \otimes h \otimes 1)\Delta^{(2)}(a) = (1 \otimes h \otimes 1)\Delta_{13}(a)$. Therefore

$$\begin{aligned}\Delta_{23}(h)\Delta^{(3)}(h) &= (\iota \otimes \Delta \otimes \iota)(1 \otimes h \otimes 1)(\Delta_{13}(h)) \\ &= \Delta_{23}(h)\Delta_{14}(h).\end{aligned}$$

From this formula we can obtain the following.

3.16 Proposition We have

$$\Delta_{23}(h)\Delta_{14}(h)\Delta_{12}(h) = \Delta_{23}(h)\Delta_{14}(h)\Delta_{34}(h).$$

Proof : We get on the one side

$$\begin{aligned}\Delta_{23}(h)\Delta_{14}(h)\Delta_{12}(h) &= \Delta_{23}(h)\Delta^{(3)}(h)\Delta_{12}(h) \\ &= \Delta_{23}(h)(\Delta \otimes \iota \otimes \iota)(\Delta^{(2)}(h)(h \otimes 1 \otimes 1)) \\ &= \Delta_{23}(h)(\Delta \otimes \iota \otimes \iota)(h \otimes \Delta(h)) \\ &= \Delta_{23}(h)(\Delta(h) \otimes \Delta(h)).\end{aligned}$$

On the other side we get

$$\begin{aligned}\Delta_{23}(h)\Delta_{14}(h)\Delta_{34}(h) &= \Delta_{23}(h)\Delta^{(3)}(h)\Delta_{34}(h) \\ &= \Delta_{23}(h)(\iota \otimes \iota \otimes \Delta)(\Delta^{(2)}(h)(1 \otimes 1 \otimes h)) \\ &= \Delta_{23}(h)(\iota \otimes \iota \otimes \Delta)(\Delta(h) \otimes h) \\ &= \Delta_{23}(h)(\Delta(h) \otimes \Delta(h)).\end{aligned}$$

This proves the formula.

3.17 Proposition The two ideals I and J are equal.

Proof : Consider the formula in the previous proposition. Apply $\iota \otimes \iota \otimes \omega \otimes \omega_1$ and assume that $\omega | J = 0$. Because of the factor $\Delta_{34}(h)$ in the right hand side of this equation, the result will be 0. So we obtain $(a \otimes b)\Delta(h) = 0$ where $a = (\iota \otimes \omega_1)\Delta(h)$ and $b = (\iota \otimes \omega)\Delta(h)$. Because this is true for all ω_1, we have that this is also true for all $a \in J$. Hence $(1 \otimes b)\Delta(h) = 0$. By our injectivity assumption, this implies $b = 0$. Because $b = (\iota \otimes \omega)\Delta(h)$, we will have that $\omega | I = 0$. This implies that $I \subseteq J$.

We have shown that the ideal I is essential. Because $I \subseteq J$, this is also the case for J. This is equivalent with the other injectivity property

$$\Delta(h)(a \otimes 1) = 0 \Rightarrow a = 0.$$

Then, by symmetry, or by a similar argument, using the factor $\Delta_{12}(h)$ on the left side, we get that also $J \subseteq I$.

We see above that one injectivity assumption gives the other. This was also true in the abelian case (see section 1). For the discrete case, this implies that also J_0 is dense in A and in fact, that $\mathcal{A} \subseteq J_0$.

It is more or less clear from the definitions that $\Delta(h)(1 \otimes A) \subseteq J \overline{\otimes} J$ and that $\Delta(h)(A \otimes 1) \subseteq J \overline{\otimes} J$. Also $\Delta(h) \in M(J \overline{\otimes} J)$. Because $\Delta(h)(1 \otimes h) = h \otimes h$, it follows that $h \in J$.

3.18 Proposition We have that $\Delta(J)(J \otimes 1)$ and $\Delta(J)(1 \otimes J)$ are dense subsets of $J \overline{\otimes} J$.

Proof : We first show that $\Delta(J)(J \otimes 1) \subseteq J \overline{\otimes} J$. For all ω and ω' we have

$$(\iota \otimes \omega')\Delta((\iota \otimes \omega)\Delta(h)) = (\iota \otimes \omega'\omega)\Delta(h) \in J.$$

Recall that $\omega'\omega$ is defined by $(\omega'\omega)(a) = (\omega' \otimes \omega)\Delta(a)$. We see that $(\iota \otimes \omega')\Delta(J) \subseteq J$ for all $\omega' \in A^*$. This implies that $\Delta(J)(1 \otimes J) \subseteq J \overline{\otimes} J$. Similarly $\Delta(J)(J \otimes 1) \subseteq J \overline{\otimes} J$.

Now we show that $\Delta(J)(J \otimes 1)$ is dense in $J \overline{\otimes} J$. Take $a \in J$. We have $\Delta(a)(1 \otimes h) = a \otimes h$ and hence $\Delta^{(2)}(a)(1 \otimes \Delta(h)) = a \otimes \Delta(h)$. If we apply $\iota \otimes \iota \otimes \omega$ we find, with $b = (\iota \otimes \omega)\Delta(h)$, that

$$a \otimes b = (\iota \otimes \iota \otimes \omega)(\Delta^{(2)}(a)(1 \otimes \Delta(h))).$$

We claim that this last element can be approximated by elements in $\Delta(J)(1 \otimes J)$. To see this, we work in the universal representation. Then $\omega = \omega_{\xi,\eta}$ for some vectors ξ and η in the underlying Hilbert space. If $(e_i)_i$ is an orthonormal basis, then

$$(\iota \otimes \iota \otimes \omega)(\Delta^{(2)}(a)(1 \otimes \Delta(h))) = \sum_i ((\iota \otimes \iota \otimes \omega_{e_i,\eta})\Delta^{(2)}(a))((\iota \otimes \iota \otimes \omega_{\xi,e_i})(1 \otimes \Delta(h))).$$

This sum converges in norm. Now, these terms in this sum are elements in $\Delta(J)(1 \otimes J)$. This gives the density of $\Delta(J)(1 \otimes J)$ in $J \overline{\otimes} J$. The other density is proved in a similar way.

4. The antipode in quasi-discrete quantum groups

In this section, the pair (A, Δ) is a quasi-discrete quantum group and h denotes the unique projection such that $\Delta(a)(1 \otimes h) = a \otimes h$ for all h. We know that also $\Delta(a)(h \otimes 1) = h \otimes a$ for all a. By the injectivity assumptions $\Delta(h)(1 \otimes a) = 0$ implies $a = 0$ and $\Delta(h)(a \otimes 1) = 0$ implies $a = 0$, we know that, in a way, $\Delta(h)$ carries enough information.

We now want to define the antipode. In the case of a Hopf algebra, we have

$$a \otimes 1 = \sum_{(a)} a_{(1)} \otimes a_{(2)} S(a_{(3)})$$

where we use the common notational convention. So, if such an element h exists, we get

$$\begin{aligned}
\Delta(h)(a \otimes 1) &= \sum_{(a)} \Delta(h)\Delta(a_{(1)})(1 \otimes S(a_{(2)})) \\
&= \sum_{(a)} \Delta(ha_{(1)})(1 \otimes S(a_{(2)})) \\
&= \sum_{(a)} \epsilon(a_{(1)})\Delta(h)(1 \otimes S(a_{(2)})) \\
&= \Delta(h)(1 \otimes S(a)).
\end{aligned}$$

We will use this formula to define S. Recall that $\Delta(h)(a \otimes 1) \subseteq (\Delta(h)(1 \otimes A))^-$ when we have a quasi-discrete quantum group (Proposition 3.12).

4.1 Definition Define the antipode S on A by

$$\mathcal{D}(S) = \{a \in A \mid \exists b \in A \;\text{ such that }\; \Delta(h)(a \otimes 1) = \Delta(h)(1 \otimes b)\}$$

and $S(a)$ by $\Delta(h)(a \otimes 1) = \Delta(h)(1 \otimes S(a))$.

Remark that the element b above is unique because of our injectivity assumptions. So S is well defined. By the same argument S is injective.

Because $\Delta(h)(h \otimes 1) = h \otimes h = \Delta(h)(1 \otimes h)$, it is immediately clear from the definition that $h \in \mathcal{D}(S)$ and that $S(h) = h$. On the other hand, it is not clear if there are enough elements in $\mathcal{D}(S)$. We will show later that this is the case. Now we prove some properties of S that follow easily from the definition.

4.2 Proposition S is closed.

Proof : Let (a_n) be a sequence in $\mathcal{D}(S)$ and assume that $a_n \to a$ and $S(a_n) \to b$ with $a, b \in A$. Then $\Delta(h)(a_n \otimes 1) = \Delta(h)(1 \otimes S(a_n))$ for all n and in the limit we get $\Delta(h)(a \otimes 1) = \Delta(h)(1 \otimes b)$. This shows that $a \in \mathcal{D}(S)$ and that $b = S(a)$. So S is closed.

4.3 Proposition If $a, b \in \mathcal{D}(S)$, then $ab \in \mathcal{D}(S)$ and $S(ab) = S(b)S(a)$.

Proof : If $a, b \in \mathcal{D}(S)$ we have

$$\begin{aligned}\Delta(h)(ab \otimes 1) &= \Delta(h)(a \otimes 1)(b \otimes 1)\\ &= \Delta(h)(1 \otimes S(a))(b \otimes 1)\\ &= \Delta(h)(b \otimes 1)(1 \otimes S(a))\\ &= \Delta(h)(1 \otimes S(b)S(a)).\end{aligned}$$

This shows that $ab \in \mathcal{D}(S)$ and that $S(ab) = S(b)S(a)$.

We want to show that $S(a)^* \in \mathcal{D}(S)$ when $a \in \mathcal{D}(S)$ and that $S(S(a)^*)^* = a$. Before we can do this, we need two more results.

4.4 Proposition For all $a \in A$ we have

$$(\Delta(h) \otimes 1)(a \otimes 1 \otimes 1)(1 \otimes \Delta(h)) = (\Delta(h) \otimes 1)(1 \otimes 1 \otimes a)(1 \otimes \Delta(h)).$$

Proof : Given $a \in A$ we have

$$\begin{aligned}(\Delta(h) \otimes 1)(a \otimes 1 \otimes 1)(1 \otimes \Delta(h)) &= (\Delta(h) \otimes 1)(\iota \otimes \Delta)(a \otimes h)\\ &= (\Delta(h) \otimes 1)(\iota \otimes \Delta)(\Delta(a)(1 \otimes h))\\ &= (\Delta(h) \otimes 1)(\Delta \otimes \iota)(\Delta(a))(1 \otimes \Delta(h))\\ &= (\Delta \otimes \iota)((h \otimes 1)\Delta(a))(1 \otimes \Delta(h))\\ &= (\Delta \otimes \iota)(h \otimes a)(1 \otimes \Delta(h))\\ &= (\Delta(h) \otimes 1)(1 \otimes 1 \otimes a)(1 \otimes \Delta(h)).\end{aligned}$$

4.5 Proposition If $a \in \mathcal{D}(S)$ then also

$$(1 \otimes a)\Delta(h) = (S(a) \otimes 1)\Delta(h).$$

Proof : By the previous proposition we know that

$$(\Delta(h) \otimes 1)(1 \otimes S(a) \otimes 1)(1 \otimes \Delta(h)) = (\Delta(h) \otimes 1)(1 \otimes 1 \otimes a)(1 \otimes \Delta(h)).$$

If we apply $\iota \otimes \iota \otimes \omega$ with $\omega \in A^*$ we get

$$\Delta(h)(1 \otimes x) = \Delta(h)(1 \otimes y)$$

with $x = (\iota \otimes \omega)(S(a) \otimes 1)\Delta(h))$ and $y = (\iota \otimes \omega)((1 \otimes a)\Delta(h))$. By the injectivity assumption, we have $x = y$. And since this holds for all ω, we get

$$(S(a) \otimes 1)\Delta(h) = (1 \otimes a)\Delta(h).$$

We can verify this formula in the case of a Hopf algebra. Indeed

$$\begin{aligned}\sum_{(a)}(S(a_{(1)})a_{(2)}\otimes a_{(3)})\Delta(h) &= \sum_{(a)}(S(a_{(1)})\otimes 1)\Delta(a_{(2)}h)\\ &= \sum_{(a)}(\epsilon(a_{(2)})S(a_{(1)})\otimes 1)\Delta(h)\\ &= (S(a)\otimes 1)\Delta(h).\end{aligned}$$

On the other hand, this expression is equal to

$$\sum_{(a)}(\epsilon(a_{(1)})1\otimes a_{(2)})\Delta(h) = (1\otimes a)\Delta(h).$$

From the formula in 4.5 we can easily proof the formula $S(S(a)^*)^* = a$.

4.6 Proposition If $a \in \mathcal{D}(S)$ then $S(a)^* \in \mathcal{D}(S)$ and $S(S(a)^*)^* = a$.

Proof : For $a \in \mathcal{D}(S)$ we have

$$(1\otimes a)\Delta(h) = (S(a)\otimes 1)\Delta(h).$$

If we take adjoints, we obtain

$$\Delta(h)(S(a)^*\otimes 1) = \Delta(h)(1\otimes a^*).$$

This shows that $S(a)^* \in \mathcal{D}(S)$ and that $S(S(a)^*) = a^*$.

Our next objective is to try to prove the (equivalent of) the formula

$$\Delta(S(a)) = \sigma(S\otimes S)\Delta(a),$$

where σ is the flip. This will follow from the formula in proposition 3.15.

4.7 Proposition If $a \in \mathcal{D}(S)$, then

$$\Delta_{23}(h)\Delta_{14}(h)(\Delta(a)\otimes 1\otimes 1) = \Delta_{23}(h)\Delta_{14}(h)(1\otimes 1\otimes \Delta(S(a))).$$

Proof : If $a \in \mathcal{D}(S)$, then

$$\begin{aligned}\Delta_{23}(h)\Delta_{14}(h)(\Delta(a)\otimes 1\otimes 1) &= \Delta_{23}(h)\Delta^{(3)}(h)(\Delta\otimes\Delta)(a\otimes 1)\\ &= \Delta_{23}(h)(\Delta\otimes\Delta)(\Delta(h)(a\otimes 1))\\ &= \Delta_{23}(h)(\Delta\otimes\Delta)(\Delta(h)(1\otimes S(a)))\\ &= \Delta_{23}(h)\Delta^{(3)}(h)(1\otimes 1\otimes \Delta(S(a)))\\ &= \Delta_{23}(h)\Delta_{14}(h)(1\otimes 1\otimes \Delta(S(a))).\end{aligned}$$

We can rewrite this formula as

$$(\Delta(h) \otimes \Delta(h))(\Delta_{13}(a)) = (\Delta(h) \otimes \Delta(h))\Delta_{42}(S(a))$$

using the right permutation. Hence we see that this formula means $(S \otimes S)\Delta(a) = \sigma\Delta(S(a))$.

Now, recall the definitions of I_0 and J_0 (see 3.11). We had

$$\begin{aligned} I_0 &= \{(\omega \otimes \iota)\Delta(h) \mid \omega \in A^*\} \\ J_0 &= \{(\iota \otimes \omega)\Delta(h) \mid \omega \in A^*\}. \end{aligned}$$

We also defined the closures $I = \overline{I_0}$ and $J = \overline{J_0}$ and we saw that $I = J$ and that I is an essential ideal of A.

We will now prove some properties of $\mathcal{D}(S)$ in connection with I_0 and J_0.

4.8 Proposition Let $\omega \in A^*$ and assume that $\omega(a) = \psi(\Delta(h)(a \otimes 1))$ for some $\psi \in (A\overline{\otimes}A)^*$. Then $(\iota \otimes \omega)\Delta(h) \in \mathcal{D}(S) \cap J_0$ and $S((\iota \otimes \omega)\Delta(h)) = (\omega_1 \otimes \iota)\Delta(h)$ where $\omega_1(a) = \psi(\Delta(h)(1 \otimes a))$.

Proof : Let ω, ω_1 and ψ be as in the formulation of the proposition. Let $a = (\iota \otimes \omega)\Delta(h)$ and $b = (\omega_1 \otimes \iota)\Delta(h)$. Then

$$\begin{aligned} \Delta(h)(a \otimes 1) &= \Delta(h)((\iota \otimes \omega)\Delta(h) \otimes 1) \\ &= (\iota \otimes \omega \otimes \iota)(\Delta_{13}(h)\Delta_{12}(h)) \\ &= (\iota \otimes \psi \otimes \iota)(\Delta_{23}(h)\Delta_{14}(h)\Delta_{12}(h)). \end{aligned}$$

On the other hand, we get

$$\begin{aligned} \Delta(h)(1 \otimes b) &= \Delta(h)(1 \otimes (\omega_1 \otimes \iota)\Delta(h)) \\ &= (\iota \otimes \omega_1 \otimes \iota)(\Delta_{13}(h)\Delta_{23}(h)) \\ &= (\iota \otimes \psi \otimes \iota)(\Delta_{23}(h)\Delta_{14}(h)\Delta_{34}(h)). \end{aligned}$$

Then, it follows from the formula in proposition 3.16, that these two expressions are the same. Therefore $\Delta(h)(a \otimes 1) = \Delta(h)(1 \otimes b)$ and we obtain $a \in \mathcal{D}(S)$ and $S(a) = b$.

In the next proposition, we will see that there are, in a way, enough elements in $\mathcal{D}(S)$.

4.9 Proposition $\mathcal{D}(S) \cap J_0$ is dense in J_0.

Proof : We saw that $(\iota \otimes \omega)\Delta(h) \in \mathcal{D}(S) \cap J_0$ if $\omega(a) = \psi(\Delta(h)(a \otimes 1))$ for some $\psi \in (A\overline{\otimes}A)^*$. Now assume that $\varphi \in A^*$ and that $\varphi((\iota \otimes \omega)\Delta(h)) = 0$ for all such ω. Then $\omega(a) = 0$ for all such ω when $a = (\varphi \otimes \iota)\Delta(h)$. This means that $\psi(\Delta(h)(a \otimes 1)) = 0$ for all $\psi \in (A\overline{\otimes}A)^*$. Hence $\Delta(h)(a \otimes 1) = 0$ and by the injectivity assumption, $a = 0$. So we find $\varphi((\iota \otimes \omega)\Delta(h)) = \omega(a) = 0$ for all $\omega \in A^*$. This means that $\varphi|J_0 = 0$. So $\mathcal{D}(S) \cap J_0$ is dense in J_0.

It is not clear what happens outside J_0. If we look at the elements we get in $\mathcal{D}(S) \cap J_0$ and if we look at the image under S, we find elements in I_0. If we take the adjoints, we find elements in $\mathcal{D}(S) \cap I_0$. In fact, we can also show, in a similar way, that $\mathcal{D}(S) \cap I_0$ is dense in I_0.

We can also prove the following.

4.10 Proposition $J_0\mathcal{D}(S) \subseteq J_0$ and $\mathcal{D}(S)I_0 \subseteq I_0$.

Proof : Because $\Delta(h)(a \otimes 1) = \Delta(h)(1 \otimes S(a))$ when $a \in \mathcal{D}(S)$ we get

$$((\iota \otimes \omega)\Delta(h))a = (\iota \otimes \omega_1)(\Delta(h))$$

where $\omega_1(x) = \omega(xS(a))$. This proves that $J_0\mathcal{D}(S) \subseteq J_0$. The other result is proved by using $(1 \otimes a)\Delta(h) = (S(a) \otimes 1)\Delta(h)$.

4.11 Proposition $\mathcal{D}(S) \cap J_0$ is a subalgebra of $\mathcal{D}(S)$.

Proof :

$$(\mathcal{D}(S) \cap J_0)(\mathcal{D}(S) \cap J_0) \subseteq \mathcal{D}(S)\mathcal{D}(S) \subseteq \mathcal{D}(S)$$
$$(\mathcal{D}(S) \cap J_0)(\mathcal{D}(S) \cap J_0) \subseteq J_0\mathcal{D}(S) \subseteq J_0.$$

We will need some more results of this type when we treat the regular representation and the Haar measure.

Let us now discuss the discrete case. We have seen that J_0 contains all summands of A. It follows from the previous results that $\mathcal{A}$ is a subalgebra of $\mathcal{D}(S)$. By taking adjoints, it is also a subalgebra of $\mathcal{D}(S^{-1})$. In fact, from the proof of 4.8, we see that $S(\mathcal{A}) = \mathcal{A}$. This is quite normal. The pair $(\mathcal{A}, \Delta)$ is a multiplier Hopf *-algebra, the antipode exists and maps $\mathcal{A}$ onto $\mathcal{A}$. Moreover, the antipode is unique and it must coincide with the antipode that we obtain here.

In this case, we also have the following phenomenon (see also [6]). If e_α is the identity in A_α, then $e_\alpha \in \mathcal{D}(S)$ and $S(e_\alpha)$ is again a minimal central projection. Hence, it must be some $e_{\alpha'}$. By the fact that $S(S(e_\alpha)^*)^* = e_\alpha$ we find $S(e_{\alpha'}) = e_\alpha$. So, although $S^2 \neq \iota$ may occur, we do have that S is involutive on the indices and on the components of A. The dimensions of A_α and $A_{\alpha'}$ must be the same. We see further that $A_{\alpha'} = \{(\iota \otimes \omega)\Delta(h) \mid \omega \in A^*$ with support in $A_\alpha\}$. See also the remark after propositiong 3.14.

We now want to look at the adjoint of S as an operator on A^*. We have to be carefull since the domain need not be dense. Formally we must have $(S_0\omega)(a) = \omega(S(a))$ when $\omega \in \mathcal{D}(S_0)$, when we use S_0 for the operator on A^*. If ω has the form ω_1 as in 4.8., that is, if $\omega(a) = \psi(\Delta(h)(1 \otimes a))$ then $(S_0\omega)(a) = \psi(\Delta(h)(1 \otimes S(a)) = \psi(\Delta(h)(a \otimes 1))$. This suggests the following definition.

4.12 Definition Define a map S_0 on A^* by

$$\mathcal{D}(S_0) = \{\omega \in A^* \mid \exists \psi \in (A \overline{\otimes} A)^* \text{ such that } \omega(a) = \psi(\Delta(h)(1 \otimes a))\}$$

and $(S_0\omega)(a) = \psi(\Delta(h)(a \otimes 1))$.

Remark that S_0 is well-defined. If $\psi(\Delta(h)(1 \otimes a)) = 0$ for all a, then also $\psi(\Delta(h)(a \otimes 1)) = 0$ for all a because, as we saw before

$$\Delta(h)(a \otimes 1) \subseteq (\Delta(h)(1 \otimes A))^-.$$

A similar argument will give here that S_0 is injective.

4.13 Proposition $\mathcal{D}(S_0)$ is w^*-dense in A^*.

Proof : If $a \in A$ and $\omega(a) = 0$ for all $\omega \in \mathcal{D}(S_0)$, then $\psi(\Delta(h)(1 \otimes a)) = 0$ for all $\psi \in (A \overline{\otimes} A)^*$. Hence $\Delta(h)(1 \otimes a) = 0$ and $a = 0$.

From the motivation before the definition, we saw already that the operators S on A and S_0 on A^* are adjoint to each other in the sense that $(S_0\omega)(a) = \omega(S(a))$ when $a \in \mathcal{D}(S)$ and $\omega \in \mathcal{D}(S_0)$. The fact that $\mathcal{D}(S)$ is not dense in A is related with the fact that S_0 need not be closed (or closable) in A^*.

We could however restrict S to $\mathcal{D}(S) \cap J_0$ and obtain a linear map from J to J which is densily defined. Then, the adjoint would become a map from J^* to J^*. We might loose some information because $\mathcal{D}(S)$ could be larger. On the other hand, we will only work with this restriction of S.

We finish this section by showing that also S_0 is an anti-homomorphism.

4.14 Proposition If $\omega_1, \omega_2 \in \mathcal{D}(S_0)$, then $\omega_1\omega_2$, defined by $(\omega_1\omega_2)(x) = (\omega_1 \otimes \omega_2)\Delta(a)$, is also in $\mathcal{D}(S_0)$ and $S_0(\omega_1\omega_2) = (S_0(\omega_2))(S_0(\omega_1))$.

Proof : Let $\psi_1, \psi_2 \in (A \overline{\otimes} A)^*$ be given such that

$$\omega_1(a) = \psi_1(\Delta(h)(1 \otimes a))$$
$$\omega_2(a) = \psi_2(\Delta(h)(1 \otimes a))$$

for all $a \in A$. Then

$$(\omega_1\omega_2)(a) = (\omega_1 \otimes \omega_2)(\Delta(a)) = (\psi_1 \otimes \psi_2)((\Delta(h) \otimes \Delta(h))(\Delta_{24}(a))).$$

On the other hand

$$\begin{aligned}((S_0\omega_2)(S_0\omega_1))(a) &= (S_0\omega_2 \otimes S_0\omega_1)(\Delta(a)) \\ &= (\psi_2 \otimes \psi_1)((\Delta(h) \otimes \Delta(h))(\Delta_{13}(a))) \\ &= (\psi_1 \otimes \psi_2)((\Delta(h) \otimes \Delta(h))(\Delta_{31}(a)).\end{aligned}$$

We have the formula
$$\Delta_{23}(h)\Delta_{14}(h) = \Delta_{23}(h)\Delta^{(3)}(h).$$
If we use the permutation σ given by
$$\sigma(a \otimes b \otimes c \otimes d) = b \otimes c \otimes a \otimes d,$$
we see that
$$\begin{aligned}(\Delta(h) \otimes \Delta(h))\Delta_{24}(a) &= \sigma(\Delta_{23}(h)\Delta_{14}(h)\Delta_{34}(a)) \\ &= \sigma(\Delta_{23}(h)\Delta^{(3)}(h)\Delta_{34}(a)) \\ &= (\Delta(h) \otimes 1)\sigma((\Delta \otimes \Delta)(\Delta(h)(1 \otimes a))).\end{aligned}$$
So $(\omega_1\omega_2)(a) = \psi(\Delta(h)(1 \otimes a))$ when
$$\psi(x) = (\psi_1 \otimes \psi_2)(\Delta(h) \otimes 1)\sigma((\Delta \otimes \Delta)(x))$$
for $x \in A\overline{\otimes}A$. This shows already that $\omega_1\omega_2 \in \mathcal{D}(S_0)$. Similarly,
$$\begin{aligned}(\Delta(h) \otimes \Delta(h))(\Delta_{31}(a)) &= \sigma(\Delta_{23}(h)\Delta_{14}(h)\Delta_{12}(a)) \\ &= \sigma(\Delta_{23}(h)\Delta^{(3)}(h)\Delta_{12}(a)) \\ &= (\Delta(h) \otimes 1)\sigma((\Delta \otimes \Delta)(\Delta(h)(a \otimes 1)))\end{aligned}$$
so that
$$(S_0(\omega_2))(S_0(\omega_1))(a) = \psi(\Delta(h)(a \otimes 1)).$$
This proves that also $S_0(\omega_1\omega_2) = (S_0\omega_2)(S_0\omega_1)$.

5. The Haar measure

If (A, Δ) is a compact quantum group, the (right invariant) Haar measure is a positive linear functional φ on A such that $(\varphi \otimes \omega)\Delta(a) = \varphi(a)\omega(1)$ whenever $\omega \in A^*$. We would like to use this as a motivation for the definition of a Haar measure on a quasi-discrete quantum group.

So, let (A, Δ) be a quasi-discrete quantum group. In this section we will assume that A is separable for technical convenience. Now the Haar measure will be a weight φ on A. Formally, we need
$$\varphi((\iota \otimes \omega)\Delta(h)) = \varphi(h)\omega(1),$$
or if we normalise φ so that $\varphi(h) = 1$, that
$$\varphi((\iota \otimes \omega)\Delta(h)) = \omega(1)$$

for all $\omega \in A^*$. Then, if $a \in A$ and if a has the form $(\iota \otimes \omega_1)\Delta(h)$ where $\omega_1 \in A^*$, we will get, again formally,

$$\begin{aligned}
\varphi((\iota \otimes \omega)\Delta(a)) &= \varphi((\iota \otimes \omega \otimes \omega_1)(\Delta^{(2)}(h))) \\
&= \varphi((\iota \otimes \omega\omega_1)(\Delta(h))) \\
&= (\omega\omega_1)(1) = \omega(1)\omega_1(1) \\
&= \varphi((\iota \otimes \omega_1)\Delta(h)) \cdot \omega(1) \\
&= \varphi(a)\omega(1).
\end{aligned}$$

Apart from the technical problems, there are some more fundamental ones here, as we will point out below.

The first problem is a consequence of the fact that the ideal J may not be all of A. So, the element ω is not determined by the element $(\iota \otimes \omega)\Delta(h)$. However, we have the following.

5.1 Lemma If $\omega, \omega' \in A^*$ and $(\iota \otimes \omega)\Delta(h) = (\iota \otimes \omega')\Delta(h)$, then $\omega|J = \omega'|J$.

Proof : If we apply any $\omega'' \in A^*$ we get $\omega(a) = \omega'(a)$ for $a = (\omega'' \otimes \iota)\Delta(h)$. Such elements are dense in J and by continuity we have $\omega|J = \omega'|J$.

Then we can define a weight φ on A.

5.2 Definition Define $\varphi : A^+ \to [0, \infty]$ by $\varphi(x) = \|\omega|J\|$ when $x = (\iota \otimes \omega)\Delta(h)$ and $\omega \in A^*_+$ and let $\varphi(x) = \infty$ when $x \in A^+$ but not of this form.

We will prove that this is a weight. The main problem is to show that $0 \leq x \leq y$ and $y = (\iota \otimes \omega)\Delta(h)$ with $\omega \in A^*_+$ implies that also $x = (\iota \otimes \omega')\Delta(h)$ for some $\omega' \in A^*_+$. This is necessary (and in fact sufficient) to show that $\varphi(x + y) = \varphi(x) + \varphi(y)$.

Before we can prove this, we need some other results. These results will also be useful in the next section.

5.3 Proposition J_0^+ is dense in J^+.

Proof : We have seen that $\mathcal{D}(S) \cap J_0$ is dense in J_0 (proposition 4.9). We also know that $J_0\mathcal{D}(S) \subseteq J_0$ (proposition 4.10). Take any $x \in J^+$. Consider a sequence (a_n) of elements in $\mathcal{D}(S) \cap J_0$ such that $a_n \to x^{1/2}$. Then $a_n^* a_n \to x$ and

$$\begin{aligned}
a_n^* a_n &\subseteq (\mathcal{D}(S) \cap J_0)^*(\mathcal{D}(S) \cap J_0) \\
&\subseteq J_0\mathcal{D}(S) \subseteq J_0.
\end{aligned}$$

So x is the limit of elements in J_0^+.

Similarly, I_0^+ is dense in J^+.

5.4 Proposition If $\omega \in A^*$ and $(\iota \otimes \omega)\Delta(h) \geq 0$, then $\omega|J \geq 0$.

Proof : Let $\overline{\omega}(x) = \omega(x^*)^-$. Then

$$(\iota \otimes \overline{\omega})\Delta(h) = ((\iota \otimes \omega)\Delta(h))^* = (\iota \otimes \omega)\Delta(h).$$

So, if we replace ω by $\frac{1}{2}(\omega + \overline{\omega})$, we may assume that ω is self-adjoint. Then decompose $\omega = \omega^+ - \omega^-$ where ω^+ and ω^- are positive. We have $\omega \leq \omega^+$ so that

$$0 \leq (\iota \otimes \omega)\Delta(h) \leq (\iota \otimes \omega^+)\Delta(h).$$

Write $x = (\iota \otimes \omega)\Delta(h)$ and $y = (\iota \otimes \omega^+)\Delta(h)$. Put $u_n = \frac{1}{(y+\frac{1}{n})^{1/2}} \cdot x^{1/2}$. Then

$$u_n u_n^* = \frac{1}{(y+\frac{1}{n})^{1/2}} x \frac{1}{(y+\frac{1}{n})^{1/2}} \leq \frac{y}{y+\frac{1}{n}} \leq 1$$

and

$$u_n^* y u_n = x^{1/2} \frac{y}{y+\frac{1}{n}} x^{1/2} \to x$$

because

$$\begin{aligned}
\|x - u_n^* y u_n\| &= \|x^{1/2}(1 - \frac{y}{y+\frac{1}{n}})x^{1/2}\| \\
&= \frac{1}{n}\|x^{1/2}\frac{1}{y+\frac{1}{n}}x^{1/2}\| \\
&= \frac{1}{n}\|u_n^* u_n\| = \frac{1}{n}\|u_n u_n^*\| \leq \frac{1}{n}.
\end{aligned}$$

We have $x \in J_0^+$. Hence $x^{1/2} \in J$ and $u_n \in J$. Choose $v_n \in \mathcal{D}(S) \cap J_0$ such that $\|v_n - u_n\| \to 0$. Then also $v_n^* y v_n \to x$. Now we have

$$\begin{aligned}
v_n^* y v_n &= v_n^*((\iota \otimes w^+)\Delta(h))v_n \\
&= (\iota \otimes \omega^+)((v_n^* \otimes 1)\Delta(h)(v_n \otimes 1)) \\
&= (\iota \otimes \omega^+)((1 \otimes S(v_n)^*)\Delta(h)(1 \otimes S(v_n)) \\
&= (\iota \otimes \omega_n)(\Delta(h))
\end{aligned}$$

where $\omega_n(a) = \omega^+(S(v_n)^* a S(v_n))$.

If we apply $\omega' \in A^*$ we get

$$\omega'(x) = \omega((\omega' \otimes \iota)\Delta(h))$$

on the one hand and

$$\omega'(x) = \lim_n \omega'(v_n^* y v_n) = \lim_n \omega_n((\omega' \otimes 1)\Delta(h)).$$

Therefore $\omega(a) = \lim \omega_n(a)$ for all $a \in I_0$. If $a \geq 0$ then $\omega_n(a) \geq 0$ so that $\omega(a) \geq 0$. Because I_0^+ is dense in J^+ we get $\omega|J \geq 0$.

5.5 Remark If $\omega \in A^*$ and $(\iota \otimes \omega)\Delta(h) \geq 0$ then, for all $\omega' \in A^*$ with $\omega' \geq 0$ we get $\omega((\omega' \otimes \iota)\Delta(h)) \geq 0$. We know already that I_0^+ is dense in J^+. If we also knew that $\{(\omega' \otimes \iota)\Delta(h) \mid \omega' \geq 0\}$ is dense in J^+, we would obtain immediately that $\omega|J \geq 0$. But for the density of $\{(\omega' \otimes \iota)\Delta(h) \mid \omega' \geq 0\}$ in J^+, we precisely need the density of I_0^+ in J^+ and (the analogue for I_0 of) the previous proposition.

So, if we combine 5.3 and 5.4 we get that the set $\{(\iota \otimes \omega)(\Delta(h)) \mid \omega \in A_+^*\}$ is dense in J_+. We need some more results before we can prove the additive property of the Haar weight.

5.6 Proposition If $a \in \mathcal{D}(S) \cap J_0$ and $x \in J_+$ then $a^*xa \in J_0$.

Proof : We can approximate $x^{1/2}$ by a sequence (b_n) in $\mathcal{D}(S) \cap J_0$ such that $\|b_n\| \leq \|x^{1/2}\|$. Let $y_n = a^*b_n^*b_na$ for all n. Then $0 \leq y_n \leq \|x\|a^*a$. Because $\mathcal{D}(S) \cap J_0$ is a subalgebra (proposition 4.11) we have $b_na \in \mathcal{D}(S) \cap J_0$. And, as in 5.3, we know that $(\mathcal{D}(S) \cap J_0)^*(\mathcal{D}(S) \cap J_0) \subseteq J_0$. So $y_n \in J_0^+$ and $a^*a \in J_0^+$. We also have of course that $y_n \to a^*xa$.

Now let $\omega_n, \omega' \in A_+^*$ such that $y_n = (\iota \otimes \omega_n)\Delta(h)$ and $\|x\|a^*a = (\iota \otimes \omega')\Delta(h)$. Then $0 \leq y_n \leq \|x\|a^*a$ will imply $\omega_n \leq \omega'$ on J by 5.4. By compactness, we may assume that $\omega_n \to \omega$ for some $\omega \in A_+^*$ in the w^*-topology on J. So, for all $\psi \in A^*$ we get

$$\begin{aligned}\psi(y_n) &= (\psi \otimes \omega_n)\Delta(h) \\ &= \omega_n((\psi \otimes \iota)\Delta(h)).\end{aligned}$$

This converges to

$$\omega((\psi \otimes \iota)\Delta(h)) = \psi((\iota \otimes \omega)\Delta(h)).$$

On the other hand $\psi(y_n) \to \psi(a^*xa)$. It follows that $a^*xa = (\iota \otimes \omega)\Delta(h)$.

Also this result would of course follow easily if we had already the hereditary property of J_0^+. We can prove this now.

5.7 Proposition If $0 \leq x \leq y$ and $y \in J_0^+$, then $x \in J_0^+$.

Proof : Choose $\omega \in A_+^*$ such that $y = (\iota \otimes \omega)\Delta(h)$. For any $a \in \mathcal{D}(S) \cap J_0$ we have $a^*ya = (\iota \otimes \omega_a)\Delta(h)$ where $\omega_a = \omega(S(a)^* \cdot S(a))$. We also have a^*xa in J_0^+.

Now, take a of the form $a = (\iota \otimes \omega)\Delta(h)$ where $\omega(z) = \psi(\Delta(h)(z \otimes 1))$ for some $\psi \in (A\overline{\otimes}A)^*$ and all z. Then $S(a) = (\omega_1 \otimes \iota)\Delta(h)$ where $\omega_1(z) = \psi(\Delta(h)(1 \otimes z))$. In the proof of proposition 4.9, we saw that such elements are dense in J. Similarly, the elements $S(a)$ with a of this form are dense in J. If we consider an approximate identity in J and approximate these in turn by elements $S(a)$ as above, we find a sequence (a_n) of elements in $\mathcal{D}(S) \cap J_0$ such that $S(a_n) \to 1$ in the strict topology of $M(J)$. We can also assume that $\|S(a_n)\| \leq 1$ for all n.

Then, ω_{a_n} is a bounded sequence and we may assume that it converges to ω'. Then

$$a_n^*ya_n = (\iota \otimes \omega_{a_n})\Delta(h) \to (\iota \otimes \omega')\Delta(h)$$

weakly. On the other hand

$$\begin{aligned}\Delta_{13}(h)(1\otimes 1\otimes a_n^* y a_n)\Delta_{23}(h) &= \Delta_{13}(h)(S(a_n)^*\otimes S(a_n)\otimes y)\Delta_{23}(h)\\ &\to \Delta_{13}(h)(1\otimes 1\otimes y)\Delta_{23}(h).\end{aligned}$$

So we must have $y=(\iota\otimes\omega')\Delta(h)$.

Having this result we can prove the main results of this section.

5.8 Proposition φ is a faithful lower semi-continuous weight.

Proof : It is clear from the definition 5.2 that $\varphi(\lambda x)=\lambda\varphi(x)$ when $x\in A^*$ and $\lambda\geq 0$. If $x,y\in A^+$ and if $x=(\iota\otimes\omega)\Delta(h)$ and $y=(\iota\otimes\omega')\Delta(h)$ for some $\omega,\omega'\in A_+^*$, then $x+y=(\iota\otimes(\omega+\omega'))\Delta(h)$ and we have $\|(\omega+\omega')|J\|=\|\omega|J\|+\|\omega'|J\|$ so that $\varphi(x+y)=\varphi(x)+\varphi(y)$. If $x,y\in A^*$ and $\varphi(x)=\infty$ or $\varphi(y)=\infty$, then also $\varphi(x+y)=\infty$ because when $x+y\in J_0^+$, also $x,y\in J_0^+$ by proposition 5.7.

We now prove that φ is lower semi continuous. Fix any $\lambda>0$ and consider a sequence (x_n) in A^+ such that $x_n\to x$ and $\varphi(x_n)\leq\lambda$. Choose elements $\omega_n\in A_+^*$ such that $x_n=(\iota\otimes\omega_n)\Delta(h)$. Because $\varphi(x_n)\leq\lambda$ we get $\|\omega_n|J\|\leq\lambda$. By compactness we can assume that $\omega_n\to\omega$ for some $\omega\in A_+^*$ in the ω^*-topology. Then, as in 5.6, we get that $x=(\iota\otimes\omega)\Delta(h)$. So also $\varphi(x)\leq\lambda$ because $\|\omega|J\|\leq\lambda$.

If x is positive and $\varphi(x)=0$, then $x=(\iota\otimes\omega)(\Delta(h)$ for some $\omega\in A_+^*$. In this case, we must have $\omega|J=0$ and $x=0$. Hence, φ is faithful.

The weight need not be semi-finite because it is only finite on J_0^+ and this need not be dense in A^+. It is of course semi-finite in the discrete case.

Now we come to the invariance property.

5.9 Proposition If $a\in A^*$ and $\varphi(a)<\infty$ and if $\omega\in A_+^*$, then

$$\varphi((\iota\otimes\omega)\Delta(a))=\|\omega|J\|\cdot\varphi(a).$$

Proof : Let $a=(\iota\otimes\omega')\Delta(h)$. Then

$$\begin{aligned}(\iota\otimes\omega)\Delta(a) &= (\iota\otimes\omega\otimes\omega')\Delta^{(2)}(h)\\ &= (\iota\otimes\omega\omega')\Delta(h).\end{aligned}$$

So $\varphi((\iota\otimes\omega)\Delta(a))=\|\omega\omega'|J\|$. If (e_n) is an approximate identity for J, then

$$\|\omega\omega'|J\|=\lim_n(\omega\omega')(e_n)=\lim_n(\omega\otimes\omega')\Delta(e_n).$$

Because $\Delta(J)(J\otimes J)$ is dense in $J\overline{\otimes}J$, we have that still $(\Delta(e_n))_n$ is an approximate identity in $J\overline{\otimes}J$. Hence

$$\lim_n(\omega\otimes\omega')\Delta(e_n)=\|(\omega\otimes\omega')|J\overline{\otimes}J)\|=\|\omega|J\|\|\omega'|J\|.$$

Therefore

$$\begin{aligned}\varphi((\iota \otimes \omega)\Delta(a)) &= \|\omega\omega'|J\| \\ &= \|\omega|J\|\|\omega'|J\| \\ &= \|\omega|J\| \cdot \varphi(a).\end{aligned}$$

6. The regular representation

Given the Haar weight φ, we can define the associated representation. Let us recall this and see what we get here.

Denote, as usual, $\mathfrak{N} = \{a \in A \mid \varphi(a^*a) < \infty\}$. So $a \in \mathfrak{N}$ iff $a^*a = (\iota \otimes \omega)\Delta(h)$ with $\omega \in A^*_+$. The set $\mathfrak{M} = \mathfrak{N}^*\mathfrak{N}$ is in this case precisely $J_0 = \{(\iota \otimes \omega)\Delta(h) \mid \omega \in A^*\}$. We have that $\mathfrak{N}$ is a left ideal and that $\langle a, b\rangle = \varphi(b^*a)$ defines a scalar product on $\mathfrak{N}$. In this case, φ is faithful and so we do not have to divide by an ideal. Denote by $\mathcal{H}$ the Hilbert space completion of $\mathfrak{N}$ with respect to this scalar product and by $\Lambda(a)$ the image of an element a in $\mathfrak{N}$ in the Hilbert space $\mathcal{H}$. The representation π of A on $\mathcal{H}$ is given by $\pi(a)\Lambda(b) = \Lambda(ab)$ when $a \in A$ and $b \in \mathfrak{N}$. Because the Haar weight is lower semi-continuous, this representation is non-degenerate. (See e.g. [16].)

We want to prove some specific results on this representation.

6.1 Proposition The space $\mathcal{D}(S) \cap J_0$ is dense in $\mathfrak{N}$ with respect to the Hilbert space norm.

Proof : We know already that

$$(\mathcal{D}(S) \cap J_0)^*(\mathcal{D}(S) \cap J_0) \subseteq J_0$$

and so $\mathcal{D}(S) \cap J_0 \subseteq \mathfrak{N}$. The more difficult part is to prove that this is dense.

Take $x \in \mathfrak{N}$ and $a \in \mathcal{D}(S) \cap J_0$. Then $xa \in \mathfrak{N}$ because $\mathfrak{N}$ is a left ideal. If $x^*x = (\iota \otimes \omega)\Delta(h)$, then

$$\|\Lambda(x) - \Lambda(xa)\|^2 = \varphi((1-a)^*x^*x(1-a)) = \|\omega_a|J\|$$

where $\omega_a(z) = \omega((1 - S(a)^*)z(1 - S(a)))$.

Now $\omega|J$ has the form $\omega_1(\,\cdot\, p)$, where $p \in J$ (see e.g. [25]). We know that p can be approximated by elements ep with $e \in J$ and $\|e\| \leq 1$ (using an approximate identity). Also e can be approximated by elements $S(a)$ with $a \in \mathcal{D}(S) \cap J_0$ (see proposition 5.7). We can assume $\|S(a)\| \leq 1$. Hence we find $a \in \mathcal{D}(S) \cap J_0$ such that $\|(1-S(a))p\|$ is small and $\|S(a)\| \leq 1$. Then $\|\omega_a|J\|$ above is small. This shows that $\mathfrak{N}(\mathcal{D}(S) \cap J_0)$ is dense in $\mathfrak{N}$. In particular $J(\mathcal{D}(S) \cap J_0)$ is dense.

Finally, because $\mathcal{D}(S) \cap J_0$ is norm dense in J, we have $(\mathcal{D}(S) \cap J_0)(\mathcal{D}(S) \cap J_0)$ and hence $\mathcal{D}(S) \cap J_0$ is dense in $\mathfrak{N}$.

In fact, in the argument above, we see that also the smaller set, namely the elements of the form $(\iota \otimes \omega)\Delta(h)$ with $\omega(a) = \psi(\Delta(h)(a \otimes 1))$ will still work. Then $\omega \in \mathcal{D}(S_0^{-1})$.

Also remark that, not only A acts non-degenerately, but this is also true for J. This also follows from the arguments in the above proof.

In that case we have the following expression for the scalar product.

6.2 Proposition If $\omega_1, \omega_2 \in \mathcal{D}(S_0^{-1})$ and $a = (\iota \otimes \omega_1)\Delta(h)$ and $b = (\iota \otimes \omega_2)\Delta(h)$, then $\varphi(b^* a) = (\omega_2^* \omega_1)(h)$.

Proof : We have $a \in \mathcal{D}(S)$ by proposition 4.8 and

$$b^* a = ((\iota \otimes \overline{\omega}_2)\Delta(h))(a) = (\iota \otimes \overline{\omega}_2)(\Delta(h)(1 \otimes S(a)).$$

So

$$\begin{aligned}\varphi(b^* a) = \overline{\omega}_2(S(a)) &= \overline{\omega}_2((S_0^{-1}\omega_1 \otimes \iota)\Delta(h)) \\ &= (S_0^{-1}\omega_1 \otimes \overline{\omega}_2)\Delta(h) \\ &= (S_0^{-1}\omega_1 \otimes S_0^{-1}\omega_2^*)\Delta(h) \\ &= (\omega_2^* \otimes \omega_1)\Delta(h) \\ &= (\omega_2^* \omega_1)(h).\end{aligned}$$

We will use $\Gamma(\omega)$ for $\Lambda((\iota \otimes \omega)\Delta(h))$. Then we get a simple expression for the representation for the representation of A.

6.3 Proposition If $a \in A$ then

$$\langle \pi(a)\Gamma(\omega_1), \Gamma(\omega_2) \rangle = (\omega_2^* \otimes \omega_1)((a \otimes 1)\Delta(h)).$$

Proof : First, let $\omega_3 \in \mathcal{D}(S_0^{-1})$ and $a = (\iota \otimes \omega_3)\Delta(h)$ and denote $b = (\iota \otimes \omega_1)\Delta(h)$. Then $ab = (\iota \otimes \omega_3)(\Delta(h)(1 \otimes S(b))$. So

$$\langle \pi(a)\Gamma(\omega_1), \Gamma(\omega_2) \rangle = (\omega_2^* \otimes \omega_3)(\Delta(h)(1 \otimes S(b)).$$

Now, for all $x \in A$ we have

$$\begin{aligned}\omega_3(xS(b)) &= \omega_3(x(S^{-1}\omega_1 \otimes \iota)\Delta(h)) \\ &= (S^{-1}\omega_1 \otimes \omega_3)((1 \otimes x)\Delta(h)).\end{aligned}$$

If $x \in \mathcal{D}(S)$ we have

$$\begin{aligned}\omega_3(xS(b)) &= (S^{-1}\omega_1 \otimes \omega_3)((S(x) \otimes 1)\Delta(h)) \\ &= (S^{-1}\omega_1)(S(x)a).\end{aligned}$$

Hence
$$\langle \pi(a)\Gamma(\omega_1), \Gamma(\omega_2)\rangle = (\omega_2^* \otimes S^{-1}\omega_1)((\iota \otimes S)\Delta(h)(1 \otimes a)).$$
By continuity, this holds for all $a \in J$. Then, if now $a \in \mathcal{D}(S^{-1})$ we get for the left hand side of the above formula
$$(\omega_2^* \otimes \omega_1)(\iota \otimes S^{-1}(a))\Delta(h)) = (\omega_2^* \otimes \omega_1)((a \otimes 1)\Delta(h)).$$
Again by continuity, this holds for all $a \in J$. Because the representation on J is already non-degenerate, this formula also hods for $a \in A$.

Using this formula, we can verify that we have a *-representation. If $a \in \mathcal{D}(S^{-1})$ we have
$$\begin{aligned}\langle \Gamma(\omega_1), \pi(a^*)\Gamma(\omega_2)\rangle &= \langle \pi(a^*)\Gamma(\omega_2), \Gamma(\omega_1)\rangle^- \\ &= (\omega_2^* \otimes \omega_1)((a^* \otimes 1)\Delta(h))^- \\ &= (\omega_2^* \otimes \omega_1^{**})(a^* \otimes 1)\Delta(h))^- \\ &= (\omega_2 \otimes \omega_1^*)((S \otimes S)(a^* \otimes 1)\Delta(h))^*) \\ &= (\omega_2 \otimes \omega_1^*)((S^{-1}(a) \otimes 1)\sigma\Delta(h)) \\ &= (\omega_1^* \otimes \omega_2)((a \otimes 1)\Delta(h)).\end{aligned}$$
We can also check positivity. Again, if $a \in \mathcal{D}(S^{-1})$ we get
$$\begin{aligned}(\omega^* \otimes \omega)((a^*a \otimes 1)\Delta(h)) &= (\omega^* \otimes \omega)(a^* \otimes S^{-1}(a))\Delta(h)) \\ &= \omega^*(a^* \cdot\,) \cdot \omega(S^{-1}(a) \cdot\,)(h) \\ &= \omega(S(a^*)^*S(\,\cdot\,)^*)^-\omega(S^{-1}(a) \cdot\,)(h) \\ &= \omega(S^{-1}(a)S(\,\cdot\,)^*)^-\omega(S^{-1}(a) \cdot\,)(h).\end{aligned}$$
This is positive because it is of the form $\langle \Gamma(\omega'), \Gamma(\omega')\rangle$ with $\omega'(z) = \omega(S^{-1}(a)z)$.

Now we want to define the associated representation of A^* by
$$\pi(\omega)\Lambda(a) = \Lambda((\iota \otimes \omega)\Delta(a)).$$
For this we need the following lemma.

6.4 Lemma If $a \in \mathfrak{N}$ and $\omega \in A^*$, then $(\iota \otimes \omega)\Delta(a) \in \mathfrak{N}$.

Proof : Let $\omega \in A^*$. Consider A in its universal representation. Consider the polar decomposition of ω on A^{**}. So $\omega = |\omega|(u\,\cdot)$ for some partial isometry. Let ξ be a vector such that $|\omega| = \langle\,\cdot\,\xi, \xi\rangle$. Then $\omega = \langle\,\cdot\,\xi, u^*\xi\rangle$.

Let $a \in \mathfrak{N}$ and put $b = (\iota \otimes \omega)\Delta(a)$. Let η be any vector and $(e_i)_{i\in I}$ a basis in the Hilbert space on which A acts. Then
$$\begin{aligned}\langle b^*b\eta, \eta\rangle = \sum_i |\langle b\eta, e_i\rangle|^2 &= \sum_i |\langle \Delta(a)\eta \otimes \xi, e_i \otimes u^*\xi\rangle|^2 \\ &\leq \|u^*\xi\|^2\|\Delta(a)\eta \otimes \xi\|^2 \\ &\leq \|\xi\|^2\langle \Delta(a^*a)\eta \otimes \xi, \eta \otimes \xi\rangle \\ &= \|\xi\|^2\langle (\iota \otimes |\omega|)\Delta(a^*a)\eta, \eta\rangle.\end{aligned}$$

So

$$b^*b \leq \|\omega\|(\iota \otimes |\omega|)\Delta(a^*a).$$

We know that $a^*a \in J_0$ and so also $(\iota \otimes |\omega|)\Delta(a^*a) \in J_0$.

Then $b^*b \in J_0$ and $b \in \mathfrak{N}$.

If we apply φ to the above equation, we get

$$\varphi(b^*b) \leq \|\omega\|\varphi((\iota \otimes |\omega|)\Delta(a^*a)) = \|\omega\|\|\omega\|\varphi(a^*a).$$

So we can define a bounded operator on $\mathcal{H}$.

6.5 Definition If $\omega \in A^*$ we define $\pi(\omega)$ on $\mathcal{H}$ by $\pi(\omega)\Lambda(a) = \Lambda(b)$ where $a \in \mathfrak{N}$ and $b = (\iota \otimes \omega)\Delta(a)$.

We see that $\|\pi(\omega)\| \leq \|\omega\|$. We also have $\pi(\omega_1\omega_2) = \pi(\omega_1)\pi(\omega_2)$ when $\omega_1, \omega_2 \in A^*$ and $\omega_1\omega_2$ is defined as before by $(\omega_1\omega_2)(a) = (\omega_1 \otimes \omega_2)\Delta(a)$. We also have that $\pi(\omega^*) = \pi(\omega)^*$ when $\omega \in \mathcal{D}(S_0^{-1})$. This will follow from the following lemma.

6.6 Lemma If $\omega, \omega_1 \in \mathcal{D}(S_0^{-1})$, then $\pi(\omega)\Gamma(\omega_1) = \Gamma(\omega\omega_1)$.

Proof : Recall that $\Gamma(\omega_1) = \Lambda(a)$ when $a = (\iota \otimes \omega_1)\Delta(h)$. So $\pi(\omega)\Gamma(\omega_1) = \Lambda(b)$ where $b = (\iota \otimes \omega)\Delta(\iota \otimes \omega_1)\Delta(h)) = (\iota \otimes \omega\omega_1)\Delta(h)$. Hence $\pi(\omega)\Gamma(\omega_1) = \Gamma(\omega\omega_1)$.

We have seen that $\langle\Gamma(\omega_1), \Gamma(\omega_2)\rangle = (\omega_2^*\omega_1)(h)$. So $\langle\pi(\omega)\Gamma(\omega_1), \Gamma(\omega_2)\rangle = (\omega_2^*(\omega\omega_1))(h) = ((\omega^*\omega_2)^*\omega_1)(h) = \langle\Gamma(\omega_1), \pi(\omega^*)\Gamma(\omega_2)\rangle$. So π is a *-representation.

This specific combination of the representation of A and of A^* will give us the fundamental unitary in the next section.

7. The fundamental unitary

We have a representation of A and A^* on $\mathcal{H}$. This gives us a representation of $A^* \otimes A$ on $\mathcal{H}\overline{\otimes}\mathcal{H}$. If $\omega \in A^*$ and $a \in A$ we can write

$$\begin{aligned}\langle\pi(\omega) \otimes \pi(a)\xi_1 \otimes \eta_1, \xi_2 \otimes \eta_2\rangle &= \langle\pi(\omega)\xi_1, \xi_2\rangle\langle\pi(a)\eta_1, \eta_2\rangle \\ &= \langle\pi(\gamma)\xi_1, \xi_2\rangle\end{aligned}$$

where $\gamma(b) = \langle\pi(a)\eta_1, \eta_2\rangle\omega(b) = \langle\pi(w(b)a)\eta_1, \eta_2\rangle$ for all $b \in A$. If we consider $\omega \otimes a$ as the linear map $f : b \to \omega(b)a$ from A to A we find

$$\gamma = \omega_{\eta_1,\eta_2} \circ \pi \circ f.$$

With $f = \iota$ we will obtain the fundamental operator W. So, we should have

$$\langle W\xi_1 \otimes \eta_1, \xi_2 \otimes \eta_2\rangle = \langle \pi(\omega_{\eta_1,\eta_2})\xi_1, \xi_2\rangle.$$

We first prove that we can define a bounded operator by this formula.

7.1 Proposition There is a bounded operator W on $\mathcal{H} \otimes \mathcal{H}$ defined by

$$\langle W\xi_1 \otimes \eta_1, \xi_2 \otimes \eta_2\rangle = \langle \pi(\omega_{\eta_1,\eta_2})\xi_1, \xi_2\rangle.$$

Proof : Take a vector $\sum_k \xi_k \otimes \eta_k$ in the algebraic tensor product of $\mathcal{H}$ with itself. Let $(e_i)_i$ be an orthonormal basis in $\mathcal{H}$.

Consider the numbers $(p_{ij})_{i,j}$ defined by

$$p_{ij} = \sum_k \langle \pi(\omega_{\eta_k,e_i})\xi_k, e_j\rangle.$$

Then

$$\begin{aligned}\sum_j |p_{ij}|^2 &= \sum_{k,\ell,j} \langle \pi(\omega_{\eta_\ell,e_i})\xi_\ell, e_j\rangle^- \langle \pi(\omega_{\eta_k,e_i})\xi_k, e_j\rangle \\ &= \sum_{k,\ell} \langle \pi(\omega_{\eta_k,e_i})\xi_k, \pi(\omega_{\eta_\ell,e_i})\xi_\ell\rangle.\end{aligned}$$

Now assume that $\xi_k = \Lambda(a_k)$ with $a_k \in \mathfrak{N}$. Then

$$\begin{aligned}\langle \pi(\omega_{n_k,e_i})\xi_k, \pi(\omega_{\eta_\ell,e_i})\xi_\ell\rangle &= \langle \Lambda((\iota \otimes \omega_{\eta_k,e_i})\Delta(a_k)), \Lambda((\iota \otimes \omega_{\eta_\ell,e_i})\Delta(a_\ell))\rangle \\ &= \varphi(((\iota \otimes \omega_{\eta_\ell,e_i})\Delta(a_\ell))^*(\iota \otimes \omega_{\eta_k,e_i})\Delta(a_k)).\end{aligned}$$

If p_i is the projection on the one-dimensional subspace $\mathbb{C}e_i$, we can rewrite this as

$$\varphi((\iota \otimes \omega_{\eta_k,\eta_\ell})(\Delta(a_\ell^*)(1 \otimes p_i)\Delta(a_k))).$$

Any finite sum over i will remain smaller than

$$\varphi(\sum_{k,\ell}(\iota \otimes \omega_{\eta_k,\eta_\ell})\Delta(a_\ell^* a_k)) = \sum_{k,\ell} \varphi(a_\ell^* a_k)\omega_{\eta_k,\eta_\ell}(1) = \| \sum_k \xi_k \otimes \eta_k\|^2.$$

This shows also that $\|W\| \leq 1$. In the following proposition, we show that $W^*W = 1$.

7.2 Proposition $W^*W = 1$.

Proof : When we look at the proof of the previous proposition, we must show that

$$\sum_i \sum_{k,\ell} (\iota \otimes \omega_{\eta_k,\eta_\ell})(\Delta(a_\ell^*)(1 \otimes p_i)\Delta(a_k))$$

converges in norm to

$$\sum_{k,\ell}(\iota\otimes\omega_{\eta_k,\eta_\ell})\Delta(a_\ell^*a_k).$$

Then we can use the lower semi-continuity of φ.

We know that $\sum p_i = 1$ in the weak operator topology. Therefore, it is sufficient to show that the map

$$x \to (\iota\otimes\omega_{\xi,\eta})(\Delta(b)(1\otimes x)\Delta(c))$$

is continuous from $\mathcal{B}(\mathcal{H})$ with the weak operator topology to A with the norm topology. We can work on the unit ball of $\mathcal{B}(\mathcal{H})$.

If $\xi = p\xi_1$ and $\eta = q\eta_1$ with $p, q \in A$, we have

$$(\iota\otimes\omega_{\xi,\eta})(\Delta(b)(1\otimes x)\Delta(c)) = (\iota\otimes\omega_{\xi_1,\eta_1})((1\otimes q^*)\Delta(b)(1\otimes x)\Delta(c)(1\otimes p)).$$

Now $\Delta(c)(1\otimes p)$ and $(1\otimes q^*)\Delta(b)$ can be approximated in norm by elements in the algebraic tensor product of A with itself. But clearly, if $p_1, q_1, p_2, q_2 \in A$, then

$$x \to (\iota\otimes\omega_{\xi_1,\eta_1})((p_1\otimes q_1)(1\otimes x)(p_2\otimes q_2))$$

has the correct continuity property. And since we work on the unit ball of $\mathcal{B}(\mathcal{H})$, this will remain true in the limit.

We can not use the same argument to show that also $WW^* = 1$ so that W is a unitary. This is of a different nature. The properties of the antipode are needed here. We will first prove the other properties of W. Then the unitarity will follow easily.

In the following proposition, we will assume that A acts on $\mathcal{H}$. Then $M(A\overline{\otimes}A)$ acts on $\mathcal{H}\overline{\otimes}\mathcal{H}$. So we can avoid the use of too many π's.

7.3 Proposition For any $a \in A$ we get $W(a\otimes 1) = \Delta(a)W$.

Proof : Take $\omega_1, \omega_2 \in A^*$ and $a \in \mathcal{D}(S^{-1})$, $\omega_2 \in \mathcal{D}(S^{-1})$. Then

$$\begin{aligned}
\langle W(a\Gamma(\omega_1)\otimes\xi)\rangle, \Gamma(\omega_2)\otimes\eta\rangle &= \langle\pi(\omega_{\xi,\eta})a\Gamma(\omega_1), \Gamma(\omega_2)\rangle \\
&= \langle\pi(\omega_{\xi,\eta})\Gamma(\omega_1(S^{-1}(a)\cdot)), \Gamma(\omega_2)\rangle \\
&= (\omega_2^*\otimes\omega_{\xi,\eta}\otimes\omega_1(S^{-1}(a)\cdot))\Delta^{(2)}(h) \\
&= ((\omega_2^*\omega_{\xi,\eta})\otimes\omega_1)(1\otimes S^{-1}(a))\Delta(h)) \\
&= ((\omega_2^*\omega_{\xi,\eta})\otimes\omega_1)((a\otimes 1)\Delta(h)) \\
&= (\omega_2^*\otimes\omega_{\xi,\eta}\otimes\omega_1)((\Delta(a)\otimes 1)\Delta^2(h)).
\end{aligned}$$

On the other hand, if $p^* \in \mathcal{D}(S^{-1})$ and $q \in A$, then

$$\begin{aligned}
\langle(p\otimes q)W(\Gamma(\omega_1)\otimes\xi), \Gamma(\omega_2)\otimes\eta\rangle &= \langle W(\Gamma(\omega_1)\otimes\xi), p^*\Gamma(\omega_2)\otimes q^*\eta\rangle \\
&= \langle\pi(\omega_{\xi,q^*\eta})\Gamma(\omega_1), \Gamma(\omega_2(S^{-1}(p^*)\cdot))\rangle \\
&= ((\omega_2(S^{-1}(p^*)\cdot))^*\otimes\omega_{\xi,q^*\eta}\otimes\omega_1)(\Delta^{(2)}(h)).
\end{aligned}$$

Now

$$
\begin{aligned}
(\omega_2(S^{-1}(p^*)\cdot)^*)(a) &= \omega_2(S(p)^*S(a)^*)^- \\
&= \omega_2(S(pa)^*)^- = \omega_2^*(pa).
\end{aligned}
$$

So

$$
\begin{aligned}
\langle (p\otimes q)W(\Gamma(\omega_1)\otimes\xi), \Gamma(\omega_2)\otimes\eta\rangle &= (\omega_2^*(p\,\cdot)\otimes\omega_{\xi,q^*\eta}\otimes\omega_1)(\Delta^{(2)}(h)) \\
&= (\omega_2^*\otimes\omega_{\xi,\eta}\otimes\omega_1)((p\otimes q\otimes 1)\Delta^{(2)}(h)).
\end{aligned}
$$

We can take $\eta = c\eta_1$ and approximate $(1\otimes c^*)\Delta(a)$ by linear combinations of $p\otimes q$. Then, by continuity we have

$$
\begin{aligned}
\langle \Delta(a)W(\Gamma(\omega_1)\otimes\xi), \Gamma(\omega_2)\otimes\eta\rangle &= (\omega_2^*\otimes\omega_{\xi,\eta}\otimes\omega_1)((\Delta(a)\otimes 1)\Delta^{(2)}(h)) \\
&= \langle W(a\Gamma(\omega_1)\otimes\xi), \Gamma(\omega_2)\otimes\eta\rangle
\end{aligned}
$$

We now want to prove $(\iota\otimes\Delta)W = W_{12}W_{13}$. We can give a meaning to this formula by applying $\omega\otimes\iota\otimes\iota$ for some $\omega\in\mathcal{B}(\mathcal{H})_*$. Indeed, $(\omega\otimes\iota)(W)\in A$. To show this, consider $\langle W\Gamma(\omega_1)\otimes\xi, \Gamma(\omega_2)\otimes\eta\rangle$. This is equal to $(\omega_2^*\otimes\omega_{\xi,\eta}\otimes\omega_1)(\Delta^2(h)) = \langle a\xi,\eta\rangle$ where $a = (\omega_2^*\otimes\iota\otimes\omega_1)(\Delta^2(h))$.

7.4 Proposition $(\iota\otimes\Delta)W = W_{12}W_{13}$.

Proof : By the remark above, we have to show $\Delta((\omega\otimes\iota)W) = (\omega\otimes\iota\otimes\iota)(W_{12}W_{13})$. We will do this for $\omega = \omega_{\xi,\eta}$. We have for $\xi,\eta,\xi_1,\eta_1,\xi_2,\eta_2\in\mathcal{H}$,

$$
\begin{aligned}
\langle \Delta((\omega_{\xi,\eta}\otimes\iota)W)\xi_1\otimes\eta_1, \xi_2\otimes\eta_2\rangle &= \langle \pi(\omega_{\xi_1,\xi_2}\omega_{\eta_1,\eta_2})\xi,\eta\rangle \\
&= \langle \pi(\omega_{\xi_1,\xi_2})\pi(\omega_{\eta_1,\eta_2})\xi,\eta\rangle \\
&= \langle W\pi(\omega_{\eta_1,\eta_2})\xi\otimes\xi_1, \eta\otimes\xi_2\rangle \\
&= \langle \pi(\omega_{\eta_1,\eta_2})\xi\otimes\xi_1, W^*(\eta\otimes\xi_2)\rangle \\
&= \langle W_{13}\xi\otimes\xi_1\otimes\eta_1, W_{12}^*(\eta\otimes\xi_2\otimes\eta_2)\rangle \\
&= \langle W_{12}W_{13}\xi\otimes\xi_1\otimes\eta_1, \eta\otimes\xi_2\otimes\eta_2\rangle.
\end{aligned}
$$

7.5 Proposition W satisfies the Pentagon equation

$$W_{23}W_{12} = W_{12}W_{13}W_{23}.$$

Proof : Applying $\omega\otimes\iota\otimes\iota$, we see that we need to prove, when $a = (\omega\otimes\iota)(W)$ that

$$W(a\otimes 1) = (\omega\otimes\iota\otimes\iota)(W_{12}W_{13})W.$$

But we have shown in 7.3 that

$$W(a \otimes 1) = \Delta(a)W$$

and in 7.4 that

$$\Delta(a) = (\omega \otimes \iota \otimes \iota)(W_{12}W_{13}).$$

Combining these results, we find the Pentagon equation.

Now, it is not so difficult anymore to show that W is a unitary.

7.6 Proposition W is unitary.

Proof : Apply $\iota \otimes \iota \otimes \omega$ to the Pentagon equation. Then

$$(1 \otimes b)W = W((\iota \otimes \iota \otimes \omega)(W_{13}W_{23}))$$

where $b = (\iota \otimes \omega)W = \pi(\omega)$. We see that $1 \otimes \pi(\omega)W$ maps the range of W into itself. Now

$$\langle W(\xi \otimes \Lambda(h)), \eta \otimes \gamma\rangle = \langle \pi(\omega_{\Lambda(h),\gamma})\xi, \eta\rangle.$$

If $\xi = \Lambda(a)$ then

$$\pi(\omega_{\Lambda(h),\gamma})\xi = \Lambda((\iota \otimes \omega_{\Lambda(h),\gamma})\Delta(a)).$$

But $\Lambda(h) = h\Lambda(h)$, so we get

$$(\Lambda(\iota \otimes \omega_{\Lambda(h)),\gamma})(\Delta(a)(1 \otimes h)) = \omega_{\Lambda(h),\gamma}(1)\Lambda(a).$$

Hence $\pi(\omega_{\Lambda(h),\gamma}) = \omega_{\lambda(h),\gamma}(1)1$. And $\langle W\xi \otimes \Lambda(h), \eta \otimes \gamma\rangle = \langle \xi, \eta\rangle\langle \Lambda(h), \gamma\rangle$. So $W\xi \otimes \Lambda(h) = \xi \otimes \Lambda(h)$. Then, also $(1 \otimes \pi(\omega)(\xi \otimes \Lambda(h))$ is in the range of W, and this is precisely $\xi \otimes \Gamma(\omega)$. These vectors span $\mathcal{H}\overline{\otimes}\mathcal{H}$.

Having this fundamental operator satisfying the Pentagon equation, one obtains the (reduced) dual by standard methods.

8. References

[1] E. Abe: *Hopf Algebras.* Cambridge University Press (1977).

[2] S. Baaj: *Répresentation régulière du groupe quantique $E_\mu(2)$ de Woronowicz.* C.R. Acad. Sci., Paris, **314** (1992), 1021–1026.

[3] S. Baaj: *Répresentation régulière du groupe quantique des déplacements de Woronowicz.* Recent advances in operator algebras (Orléans, 1992), Astérisque **232** (1995), 11–48.

[4] S. Baaj & G. Skandalis: *Unitaires Multiplicatifs et dualité pour les Produits Croisés de C^*-Algèbres.* Ann. Scient. Ec. Norm. Sup., **26** (1993), 425–488.
[5] V.G. Drinfel'd: *Quantum Groups.* Proceedings ICM Berkeley (1986), 798–820.
[6] E.G. Effros & Z.-J. Ruan: *Discrete Quantum Groups I. The Haar Measure.* Int. J. Math. **5** (1994), 681–723.
[7] M. Enock & J.-M. Schwartz: *Kac Algebras and Duality of Locally Compact Groups.* Springer, Berlin (1992).
[8] K.H. Hofmann: *Elements of compact semi-groups.* Charles E. Merill Books Inc. Columbus, Ohio (1996).
[9] J. Hollevoet & J. Quaegebeur: *Duality for Quantum Groups.* Preprint K.U.Leuven (1992). Unpublished.
[10] J. Kustermans: *Quasi-discrete quantum groups are almost discrete.* Preprint K.U.Leuven (1995). Unpublished.
[11] J. Kustermans & S. Vaes: *A simple definition for locally compact quantum groups.* C.R. Acad. Sci., Paris, Sér. I Math. **328** (1999), 871–876.
[12] J. Kustermans & S. Vaes: *Locally compact quantum groups.* Ann. Scient Éc. Norm. Sup. **33** (2000), 837–934.
[13] J. Kustermans & S. Vaes: *Locally compact quantum groups in the von Neumann algebraic setting.* Math. Scand. **92** (2003), 68–92.
[14] M.B. Landstad & A. Van Daele: *Compact and discrete subgroups of algebraic quantum groups.* Preprint University of Trondheim and K.U.Leuven (2004).
[15] B.D. Malviya & B.J. Tomiuk: *Multiplier operators on B^*-algebras.* Proc. Amer. Math. Soc. **31** (1972) 505–510.
[16] G.K. Pedersen: *C^*-algebras and their automorphism groups.* Academic Press, London (1979).
[17] P. Podleś & S.L. Woronowicz: *Quantum deformation of Lorentz group.* Commun. Math. Phys. **130** (1990), 381–431.
[18] S. Sakai: *C^*-algebras and W^*-algebras.* Springer Verlag (Berlin) 1971.
[19] G. Skandalis *Operator Algebras and Duality.* Proceedings ICM Kyoto (1990), 997–1009.
[20] M.E. Sweedler: *Hopf Algebras.* Mathematical Lecture Note Series. Benjamin. N.Y., 1969.
[21] D.C. Taylor: *The strict topology for double centralizers algebras.* Trans. Amer. Math. Soc. **150** (1970) 633–643.
[22] J.-M. Vallin: *C^*-algèbres de Hopf et C^*-algebrès de Kac.* Proc. Lond. Math. Soc. **50** (1965) 131–174.
[23] A. Van Daele: *Dual pairs of Hopf *-algebras.* Preprint K.U.Leuven (1990) (first version); Bull. Lond. Math. Soc. **25** (1993) 209–230 (second version).
[24] A. Van Daele: *Multiplier Hopf Algebras.* Trans. Amer. Math. Soc. **342** (1994), 917–932.
[25] A. Van Daele: *The Haar measure on a compact quantum group.* Proc. Amer. Math. Soc. **123** (1995), 3125–3128.
[26] A. Van Daele: *Discrete quantum groups.* J. of Alg. **180** (1996), 431–444.

[27] A. Van Daele: *The Haar measure on finite quantum groups.* Proc. Amer. Math. Soc. **125** (1997), 3489–3500.

[28] A. Van Daele & S. Wang: *The Larson-Sweedler theorem for multiplier Hopf algebras.* Preprint K.U.Leuven (2004).

[29] S.L. Woronowicz: *Pseudospaces, pseudogroups and Pontryagin duality.* Proceedings of the International Conference on Mathematics and Physics, Lausanne (1979), Lecture Notes in Physics **116** () 407–412.

[30] S.L. Woronowicz: *Compact Matrix Pseudogroups.* Comm. Math. Physics **111** (1987), 613–665.

[31] S.L. Woronowicz: *Compact quantum groups.* Symétries quantiques (Les Houches, 1995), 845–884, North Holland, Amsterdam (1998).